

ENDOSCOPIC TRANSFER OF ORBITAL INTEGRALS IN LARGE RESIDUAL CHARACTERISTIC

JULIA GORDON AND THOMAS HALES

ABSTRACT. This article constructs Shalika germs in the context of motivic integration, both for ordinary orbital integrals and κ -orbital integrals. Based on transfer principles in motivic integration and on Waldspurger's endoscopic transfer of smooth functions in characteristic zero, we deduce the endoscopic transfer of smooth functions in sufficiently large residual characteristic.

We dedicate this article to the memory of Jun-Ichi Igusa. The second author wishes to acknowledge the deep and lasting influence that Igusa's research has had on his work, starting with his work as a graduate student that used Igusa theory to study the Shalika germs of orbital integrals, and continuing today with themes in motivic integration that have been inspired by the Igusa zeta function.

This article establishes the endoscopic matching of smooth functions in sufficiently large residual characteristic. The main conclusions are based on four fundamental results: Langlands-Shelstad descent for transfer factors [26], Ngô's proof of the fundamental lemma [28], Waldspurger's proof that the fundamental lemma implies endoscopic matching of smooth functions in characteristic zero [34], and the Cluckers-Loeser version of motivic integration [12], including transfer principles for deducing results for one nonarchimedean field from another nonarchimedean field with the same residue field [13]. We use recent extensions of the transfer principle to transfer linear dependencies from one field to another [9].

We note that the term transfer is used with two separate meanings in this article. *Endoscopic matching*¹ refers to the matching of κ -orbital integrals on a reductive group with stable orbital integrals on an endoscopic group, in a form made precise by the Langlands-Shelstad transfer factor. On the other hand, *transfer principles* refer to the transfer of first-order statements or properties of constructible functions from one nonarchimedean field to another nonarchimedean field with the same residual characteristic. In this article, we will refer to endoscopic matching, transfer factors, and transfer principles.

In our main results, the constraints on the size of the residual characteristic are not effective. This means that our results are not known to apply to any particular nonarchimedean field of positive characteristic. This seems to be a serious limitation of our methods. Nonetheless, we hope that our results about the constructibility of Shalika germs can serve as a further illustration of the close connection between harmonic analysis of p -adic groups and motivic integration.

¹It is called endoscopic transfer in the literature, title, and abstract, but we prefer to call it endoscopic matching in the body of the article because of the other uses of the word transfer. We avoid the awkward but apt phrase "transfer of transfer."

0.1. Acknowledgement. J.G. is grateful to Radhika Ganapathy for suggesting this project and helpful conversations at the beginning; we also thank MSRI, where this collaboration began.

1. STATEMENT OF RESULTS

This introductory section surveys the main results of this article. Unexplained terminology and notation are explained in the main body of the article.

We use the concepts of *definable sets* and *constructible motivic functions* (or constructible functions for short) from [12]. All constructible functions in this article are understood in this sense. In fact, in [12] a closely related notion of *definable subassignment* is used instead of our notion of a *definable set*, but this distinction does not affect the contents of this article.

In this article, by nonarchimedean field we mean a non-discrete locally compact nonarchimedean valued field; that is, a field isomorphic to a finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$ for some prime p . Let Loc_m denote the set of all nonarchimedean fields whose residual characteristic is at least $m \in \mathbb{N} = \{0, 1, 2, \dots\}$. To avoid set-theoretic issues, we assume that (the carrier of) each field is a subset of some fixed cardinal number (that is sufficiently large to obtain every such field up to isomorphism). If S is a definable set, and $F \in \text{Loc}_m$, we write $S(F)$ for the interpretation of S in F . If f is a constructible function on S , we write f_F for the corresponding function $f_F : S(F) \rightarrow \mathbb{C}$.

Let G_Z be a definable reductive group over a definable cocycle space Z in the sense of [10] with a given set of fixed choices enumerated in Section 2.1. For each $k \in \mathbb{N}$, there is a definable set NF_G^k representing k -tuples of Barbasch-Moy pairs $\Upsilon = (N, \mathfrak{f})$, in the sense of Section 2.2.4, with pairwise non-conjugate nilpotent elements N in the Lie algebra \mathfrak{g} of G . Here, $\mathfrak{f} \in \mathcal{F}$, where \mathcal{F} is a parahoric indexing set.

For $F \in \text{Loc}_m$ and $z \in Z(F)$, we have F -points $G_z(F)$ of a connected reductive group over F , obtained by twisting the split group by the cocycle z . We also have F -points $\mathfrak{g}_z(F)$ of a Lie algebra. There is a definable set $\mathfrak{g}^{\text{rss}}$ of regular semisimple elements in \mathfrak{g} .

The Shalika germs of orbital integrals were first constructed in [32]. Our first theorem establishes the existence of constructible motivic functions that represent Shalika germs. The following theorem is proved in Section 5.1.

Theorem 1 (Lie algebra Shalika germs). *For each $n \in \mathbb{N}$, there exists an n -tuple Γ of constructible functions with domain $\mathfrak{g}^{\text{rss}} \times_Z \text{NF}_G^n$ and a constructible function d_n with domain NF_G^n . There exists $m \in \mathbb{N}$, such that for every field $F \in \text{Loc}_m$, and for all $z \in Z(F)$, if $k = k_{F,z}$ is the number of nilpotent orbits in $\mathfrak{g}_z(F)$, then $d_{k,F}$ does not vanish anywhere on the nonempty set $\text{NF}_{G_z}^k(F)$. Moreover, for every $\Upsilon \in \text{NF}_{G_z}^k(F)$, the function $X \mapsto \Gamma(X, \Upsilon)_{i,F}/d_{k,F}(\Upsilon)$ is the Shalika germ (for some normalization of measures) at $X \in \mathfrak{g}_z^{\text{rss}}(F)$ on the orbit of N_i , where N_i is the first component of $\Upsilon_i = (N_i, \mathfrak{f}_i)$, for $i = 1, \dots, k$.*

Although the functions are defined on the full set of regular elements on the Lie algebra \mathfrak{g} , it is only their behavior in a small neighborhood of 0 that matters. We make no claims about their behavior far from 0.

The first components of $\Upsilon_i = (N_i, \mathfrak{f}_i)$, for $i = 1, \dots, k$, give a complete non-redundant enumeration of nilpotent orbits of G_z . The theorem realizes all Shalika germs as constructible functions. In a sense that can be made precise (the residual characteristic must be large with respect to fixed choices of the field-independent data such as the root data), every large

residual characteristic reductive group can be represented as fiber G_z in a definable reductive group G/Z , so that the theorem is general in large residual characteristic.

We obtain a similar representation as constructible functions for Shalika germs in the group in terms of unipotent conjugacy classes. For each n , there is a definable set UF_G^n representing n -tuples of unipotent Barbasch-Moy pairs in G whose tuple of first coordinates are pairwise non-conjugate (Section 2.2.4). The following theorem is proved in Section 5.5.

Theorem 2. *If we replace NF_G^n (resp. $\mathrm{NF}_{G_z}^k(F)$, $\mathfrak{g}^{\mathrm{rss}}$) with UF_G^n (resp. $\mathrm{UF}_{G_z}^k(F)$, G^{rss}) in Theorem 1, the same statement holds in the group.*

Our next theorem gives a representation of the Langlands-Shelstad transfer factor as a constructible function. This result was previously known for unramified groups [11]. The proof appears in Section 3.

Theorem 3 (Constructibility of Lie algebra transfer factors). *Let $(G, H)_Z$ be a definable reductive group and associated endoscopic group over a cocycle space Z , with Lie algebras \mathfrak{g} and \mathfrak{h} . There exists a constructible function Δ on*

$$(1) \quad V := \mathfrak{h}^{G-\mathrm{rss}} \times_Z \mathfrak{g}^{\mathrm{rss}} \times_Z \mathfrak{h}^{G-\mathrm{rss}} \times_Z \mathfrak{g}^{\mathrm{rss}}$$

and a natural number $m \in \mathbb{N}$ such that for every field $F \in \mathrm{Loc}_m$, every $z \in Z(F)$, and for every

$$(X_H, X_G, \bar{X}_H, \bar{X}_G) \in V_z(F),$$

the Lie algebra Langlands-Shelstad transfer factor is $\Delta_F(X_H, X_G, \bar{X}_H, \bar{X}_G)$.

Here, $\mathfrak{h}^{G-\mathrm{rss}}$ denotes the definable set of G -regular semisimple elements in \mathfrak{h} . We are unable to obtain the Langlands-Shelstad transfer factor on the full group, because there is currently no good theory of multiplicative characters for motivic integration. The following theorem is proved in Section 5.5, by adapting the Lie algebra proof.

Theorem 4 (Constructibility of group transfer factors). *Let $(G, H)_Z$ be a definable reductive group and associated endoscopic group over a cocycle space Z . There exists a constructible function Δ on the set of G -regular elements in a definable neighborhood V of 1 in*

$$H \times_Z G \times_Z H \times_Z G$$

and a natural number $m \in \mathbb{N}$ such that for every field $F \in \mathrm{Loc}_m$, every $z \in Z(F)$, and for every

$$(\gamma_H, \gamma_G, \bar{\gamma}_H, \bar{\gamma}_G) \in V_z^{G-\mathrm{rss}}(F),$$

the Langlands-Shelstad transfer factor for the group $G_z(F)$ restricted to $V_z(F)$ is $\Delta_F(\gamma_H, \gamma_G, \bar{\gamma}_H, \bar{\gamma}_G)$.

Based on the the constructibility of Shalika germs and the transfer factor, we obtain a constructible function representing the Shalika germs of κ -orbital integrals (Equation 5). As a special case, we obtain the constructibility of stable Shalika germs (Equation 6). These constructions work both in the Lie algebra and in the group. The following theorem is proved in Section 5.5. An alternative version of the theorem for Lie algebras is proved in Section 5.4.

Theorem 5 (local endoscopic matching of orbital integrals). *Let $(G, H)_Z$ be a definable reductive group G with definable endoscopic group H over a cocycle space Z . There exists a natural number $m \in \mathbb{N}$ such that for all $F \in \mathrm{Loc}_m$, and all $z \in Z(F)$, the local endoscopic matching of orbital integrals holds for (G_z, H_z) . That is, for all $f \in C_c^\infty(G_z(F))$, there*

exists a function $f^H \in C_c^\infty(H_z(F))$ for which the κ -orbital integrals of f (with the usual Langlands-Shelstad transfer factor) are equal to the stable orbital integrals of f^H near 1. More precisely,

$$SO(\gamma_H, f^H) = \sum_{\gamma_G} \Delta_F(\gamma_H, \gamma_G, \bar{\gamma}_H, \bar{\gamma}_G) O(\gamma_G, f),$$

for all $\gamma_H \in H_z^{G-\text{rss}}(F)$ near 1.

Our wording in Theorem 5 is based on definition of *local Δ -matching*² at the identity in [24]. It can be expressed equivalently in terms of Shalika germs. The wording in Theorem 6 is based on the definition of *Δ -matching* in [24]. Recall that Δ -matching (and the definition of the Langlands-Shelstad transfer factor away from 1) requires a choice of central extension \tilde{H} of the endoscopic group H and a character θ on the center of $\tilde{H}(F)$ [26, §4.4]. The following theorem is our main result. The proof appears in Section 5.6.

Theorem 6 (endoscopic matching of orbital integrals). *Let $(G, H)_{/Z}$ be a definable reductive group G with definable endoscopic group H over a cocycle space Z . There exists a natural number $m \in \mathbb{N}$ such that for all $F \in \text{Loc}_m$, and all $z \in Z(F)$, the endoscopic matching of orbital integrals holds for (G_z, H_z) . That is, for all $f \in C_c^\infty(G_z(F))$, there exists a $f^{\tilde{H}} \in C_c^\infty(\tilde{H}_z(F), \theta)$ for which the κ -orbital integrals of f (with the usual Langlands-Shelstad transfer factor) are equal to the stable orbital integrals of $f^{\tilde{H}}$. More precisely,*

$$SO(\gamma_{\tilde{H}}, f^{\tilde{H}}) = \sum_{\gamma_G} \Delta_F(\gamma_{\tilde{H}}, \gamma_G, \bar{\gamma}_{\tilde{H}}, \bar{\gamma}_G) O(\gamma_G, f),$$

for all $\gamma_{\tilde{H}} \in \tilde{H}_z^{G-\text{rss}}(F)$.

In a sense that can be made precise (the residual characteristic must be large with respect to fixed choices of the field-independent data such as the root data), every large residual characteristic reductive group with endoscopic group can be represented as a fiber (G_z, H_z) of a definable pair $(G, H)_{/Z}$, so that the theorem is comprehensive in large residual characteristic.

We prove other theorems that we do not state in this introduction that may be of interest. We give a transfer principle for asymptotic expansions (Theorem 11) and prove a uniformity result for the asymptotics as the nonarchimedean field varies (Theorem 14). We give a classification of definable reductive groups in terms of fixed data (Section 6). This classification departs in noteworthy ways from the classification of reductive groups over nonarchimedean fields. This leads to the conclusion, for example, that nonisomorphic unitary groups of the same rank over a nonarchimedean field can have Shalika germs given by the same formula. A final section lists some open problems (Section 7).

2. REVIEW OF MOTIVIC INTEGRATION IN REPRESENTATION THEORY

In this section we review basic constructs of motivic integration, applied to representation theory and harmonic analysis, as developed in [11] and [10]. We assume familiarity with definable sets and constructible functions, as developed in [12].

We work in the Denef-Pas language, a three-sorted first-order language. The three sorts are called the valued field sort, the residue field sort, and the value group sort. The valued

²It is called local Δ -transfer in [24].

field sort and residue field sort both contain the first-order language of rings, and the value group sort contains the first-order language of ordered groups. There is function symbol ord from the valued field sort to the value group sort, and function symbol $\bar{\alpha}c$ from the valued field sort to the residue field sort, which are interpreted as the valuation map and angular component map respectively.

By a *fixed choice*, we mean a fixed set that does not depend on the the Denef-Pas language or its variables in any way. Fixed choices are assumed to be made at the outermost level, and will sometimes be dropped from the notation. Various formulas in the Denef-Pas language will be constructed from fixed choices.

A free parameter (or simply parameter) refers to a collection of free variables of the same sort in a formula in the Denef-Pas language, ranging over a definable set. A bound parameter is similar, except that the variables are all bound by a contiguous block of existential or a contiguous block of universal quantifiers. By a parameter in VF (for valued field), we mean a variable of valued field sort.

2.1. Fixed choices. We let G^{**} be a split connected reductive group over \mathbb{Q} , with Borel B^{**} and Cartan $T^{**} \subset B^{**} \subset G^{**}$ all over \mathbb{Q} . We let $(X^*, X_*, \Phi, \Phi^\vee)$ (characters, cocharacters, roots, and coroots) be the root datum for G^{**} , all with respect to (B^{**}, T^{**}) .

We let Σ be a (large) abstract finite group (that plays the role of a Galois group), with a fixed enumeration $1 = \sigma_1, \dots, \sigma_n$ of its elements, where $n = \text{card } \Sigma$. We assume that Σ comes with a fixed short exact sequence

$$1 \rightarrow \Sigma^t \rightarrow \Sigma \rightarrow \Sigma^{unr} \rightarrow 1,$$

where Σ^t (called the tame inertia) and Σ^{unr} (called the unramified quotient) are both assumed to be cyclic. We fix a generator qFr of Σ^{unr} , called the quasi-Frobenius element.

We choose an action of Σ on the root datum stabilizing the set of simple roots. Specifically, we fix a homomorphism

$$\rho_G : \Sigma \rightarrow \text{Out}(G^{**}, B^{**}, T^{**}, \{X_\alpha\}),$$

from Σ to the group of automorphisms of G^{**} fixing a pinning, which we identify with the group of outer automorphisms of G^{**} . We use the action ρ_G on the root datum to form the L -group

$${}^L G = \hat{G} \rtimes \Sigma,$$

where \hat{G} is the complex Langlands dual of G^{**} . We fix (\hat{T}, \hat{B}) dual to (T^{**}, B^{**}) . We assume that the action of Σ on \hat{G} preserves a pinning $(\hat{T}, \hat{B}, \{\hat{X}_\alpha\})$ of \hat{G} .

We choose a semisimple element $\kappa \in \hat{T}^\Sigma$, with connected centralizer $\hat{H} = C_{\hat{G}}(\kappa)^0$. We let $(X^*, X_*, \Phi_H, \Phi_H^\vee)$ be the root datum dual to that of \hat{H} . We choose as part of the endoscopic data, a homomorphism $\rho_H : \Sigma \rightarrow C_{L_G}(\kappa)/\hat{T}$, through which we construct

$${}^L H = \hat{H} \rtimes \Sigma.$$

We assume that Σ acts on \hat{H} through automorphisms that preserve some pinning of \hat{H} . This choice induces an action of Σ on the root datum (giving two actions, ρ_G and ρ_H , of Σ on X^* : one from G and one from the endoscopic data). Temporarily disregarding Σ , we let H^{**} be the split group over \mathbb{Q} with root datum $(X^*, X_*, \Phi_H, \Phi_H^\vee)$. We fix a linear algebraic subgroup $\text{Aut}_0(G^{**})$ with finitely many connected components of $\text{Aut}(G^{**})$ that contains the full group of inner automorphisms and the image of Σ .

To treat the theory of reductive groups in a definable context, we fix faithful rational representations of all split reductive groups (over \mathbb{Q}) of interest: G^{**} , G_{sc}^{**} , G_{der}^{**} , G_{adj}^{**} , $\text{Aut}_0(G^{**})$, and even $(G_{\text{sc}}^{**} \times G_{\text{sc}}^{**})/\text{Zent}_{\text{sc}}$, with diagonal embedding of the center Zent_{sc} of the simply connected cover of the derived group of G^{**} . We do the same for H^{**} and its affiliated groups. We fix the obvious morphisms ($G^{**} \rightarrow G_{\text{adj}}^{**}$, and so forth) between affiliated groups by explicit polynomial maps (expressed as collections of polynomials with rational coefficients). The rational representations of the reductive groups give representations of all of the associated Lie algebras.

The root datum and pinning for G^{**} give a Dynkin diagram $\text{Dyn}(\Phi^+(G^{**}))$ whose nodes are the simple roots of (G^{**}, B^{**}, T^{**}) . The group Σ acts on the Dynkin diagram by permutation of the nodes. Let τ be an index running over the set of connected components of the Dynkin diagram. For each connected component Dyn_τ of the diagram, we consider the subgroup Σ_τ stabilizing the component, and the kernel $\Sigma_\tau^0 \subset \Sigma_\tau$ of the action on the component. We let ℓ_τ be the number of connected components in the Σ orbit of τ . We let $d_\tau = [\Sigma_\tau : \Sigma_\tau^0]$ be the index. We add the prefix superscript d_τ to the symbol for the connected Dynkin diagram in the usual way: 1A_n , 2A_n , and so forth.

Following [17, 30], for each τ , we form an affine diagram ${}^e\mathcal{R}_\tau$ attached to the tame inertia group $\Sigma_\tau^t = \Sigma^t \cap \Sigma_\tau$. Its set of nodes is $\{0\} \cup I_\tau$, where I_τ is the set of orbits of simple roots of Dyn_τ under the action of Σ_τ^t . The element $\{0\}$ represents an extended node. The superscript prefix e is the order $[\Sigma_\tau^t : \Sigma_\tau^t \cap \Sigma_\tau^0]$ of the group of tame inertial automorphisms of Dyn_τ . We drop superscript e from the notation when $e = 1$. The various connected affine diagrams ${}^e\mathcal{R} = \mathcal{A}_n, {}^2\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \dots, \mathcal{E}_8$ are listed in a table in [17].

We set $\Sigma_\tau^{\text{unr}} = \Sigma_\tau / \Sigma_\tau^0 \Sigma_\tau^t$. For each τ , we choose an action

$$\phi_\tau : \Sigma_\tau^{\text{unr}} \rightarrow \text{Aut}({}^e\mathcal{R}_\tau)$$

of Σ_τ^{unr} on the affine diagram. When $e \neq 1$, by inspection of the various affine diagrams we see that the order of the automorphism group is at most 2, so that the choice of ϕ_τ amounts to at most a binary choice of whether the action is trivial or nontrivial. When $e = 1$, the situation is only slightly more involved.

From each orbit of Σ on the set of components of the Dynkin diagram $\text{Dyn}(\Phi^+(G^{**}))$ we choose a representative τ . Let $A = \{\tau\}_\tau$ be this set of representatives. Let S (depending on all the choices above) be the product

$$S = \prod_{\tau \in A} \text{node}({}^e\mathcal{R}_\tau) / \phi_\tau,$$

of the ϕ_τ -orbits of nodes in each affine diagram. Let $\mathcal{F} = \mathcal{F}(G^{**}, \Sigma, \rho_G, \phi_\tau)$ be the set of all subsets S' of S such that the projection of S' to each factor is nonempty. We call \mathcal{F} the parahoric indexing set.

The next subsection will give various parameter spaces that admit interpretations in nonarchimedean structures. When the fixed data becomes associated with a reductive group G over a nonarchimedean field F , we either get nothing (if our data does not satisfy required compatibility requirements) or we get the actual indexing set for the parahoric subgroups in that reductive group. This follows directly from the explicit description of parahorics in [17], which is the starting point for our definition of \mathcal{F} . Parameters $\mathfrak{f} \in \mathcal{F}$ may be identified with barycentric centers of facets in a standard alcove in a suitable apartment of the Bruhat-Tits

building. Groups and algebras in the Moy-Prasad filtration may be associated with these points in the building. By [10], for each parameter $\mathfrak{f} \in \mathcal{F}$, there are definable sets $G_{\mathfrak{f},0+}$, $\mathfrak{g}_{\mathfrak{f},0}$, $\mathfrak{g}_{\mathfrak{f},0+}$ the pro-unipotent radical of the parahoric subgroup, a parahoric subalgebra, and one of its subalgebras.

2.2. Parameter spaces. We recall our conventions for handling Galois cohomology and the theory of reductive groups in the Denef-Pas language. Further details about these constructions can be found in [11] and [10]. We describe various parameter spaces (or simply spaces for short) with variables of the valued field sort.

2.2.1. Field extensions. The parameter space of field extensions³ E/VF of a fixed degree n is defined to be the parameter space of $(a_0, \dots, a_{n-1}) \in \text{VF}^n$ (that is, n variables of the valued field sort) such that

$$p_a(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0$$

is an irreducible polynomial. Field arithmetic is expressed in terms of operations on VF^n by means of the identifications

$$E = \text{VF}[t]/(p_a(t)) \simeq \text{VF}^n.$$

The space of automorphisms of E/VF consists of linear maps (brought back to arithmetic on $\text{End}(E) = \text{End } \text{VF}^n = \text{VF}^{n^2}$) respecting the field operations. The space of Galois field extensions E/VF of fixed degree n is definable by the condition that there exist n distinct automorphisms of the field E/VF . There is a space of Galois field extensions of fixed degree n , with an enumeration of its automorphisms, given by tuples

$$(2) \quad (E, \sigma_1, \sigma_2, \dots, \sigma_n),$$

where E/VF is a Galois field extension and σ_i are the distinct field automorphisms. For fixed abstract group Σ of order n with full enumeration $\sigma'_i \in \Sigma$, for $i = 1, \dots, n$, there is a space of Galois field extensions with Galois groups isomorphic to Σ given by tuples in Formula 2 with the additional isomorphism requirement that

$$\sigma_i \sigma_j = \sigma_k \iff \sigma'_i \sigma'_j = \sigma'_k.$$

The space of unramified field extensions E/VF of a fixed degree n is a definable subspace of the space of all field extensions. There is a space of pairs (E, qFr) where E is an unramified extension and qFr is a fixed generator of the Galois group of E/VF .

There is a space of field extensions $K/E/\text{VF}$, for fixed degrees E/VF and K/VF . It is specified by irreducible polynomials p and q for $E = \text{VF}[t]/(p(t))$ and $K = \text{VF}[t']/(q(t'))$, and an element $t'' \in \text{VF}[t']/(q(t'))$ (the image of t under $E \rightarrow K$) such that $p(t'') = 0$. There is a space of Galois field extensions $K/E/\text{VF}$ for fixed degrees for E/VF and K/VF with enumerations of the Galois groups of K/VF and K/E , together with a table describing the homomorphism $\text{Gal}(K/\text{VF})$ to $\text{Gal}(E/\text{VF})$.

If we have a short exact sequence of enumerated groups

$$1 \rightarrow \Sigma^t \rightarrow \Sigma \rightarrow \Sigma^{unr} \rightarrow 1,$$

there is a space of Galois field extensions $K/E/\text{VF}$ with enumerated automorphisms $\sigma_1, \dots, \sigma_n$, such that the automorphisms in Σ^t act trivially on E , K/E is totally ramified with Galois group Σ^t , K/VF has Galois group Σ , and E/VF is unramified with Galois group Σ^{unr} .

³When we work with field extensions, we will write E/VF , rather than the more cumbersome VE/VF , etc.

2.2.2. *Galois cocycles.* The space of Galois cocycles is given as tuples

$$(K, \sigma_1, \dots, \sigma_n, a_1, \dots, a_n),$$

where $a_i \in K^k$ for some k . Here K/VF is a Galois extension of fixed degree n , with enumerated Galois group σ_i subject to the cocycle relations:

$$\sigma_i \sigma_j = \sigma_k \implies a_i \sigma_i(a_j) = a_k \quad \text{for all } i, j, k.$$

For fixed choices of G^{**} , action of Σ on the root datum, there is a space $(K, \sigma_1, \dots, a_1, \dots, G^*)$ of quasi-split reductive groups split over K with enumerated isomorphism between the Galois group of K/VF and Σ . Here the elements a_i enumerate the cocycle in the outer automorphism group (identified with automorphisms fixing a pinning $(B^{**}, T^{**}, \{X_\alpha\})$) defining the quasi-split form. The pair (B^{**}, T^{**}) gives a pair (B^*, T^*) in the quasi-split form.

G is a definable family of reductive groups parametrized by a definable cocycle space Z , which is constructed as in [10]. We include in the space Z an explicit choice of inner twisting

$$\psi : G \times_{\text{VF}} K \rightarrow G^* \times_{\text{VF}} K$$

to the quasi-split inner form that agrees with an enumerated cocycle $\sigma_i(\psi)\psi^{-1}$ with values in $G_{\text{adj}}^*(K)$, also given as part of the data of Z . In fact, we identify G with G^* over K , so that ψ is the identity, but it is retained in notation in the form of a cocycle $\sigma(\psi)\psi^{-1}$.

Let $\mathfrak{c}_{\mathfrak{g}^*} = \mathfrak{g}/G$ be Chevalley's adjoint quotient, which we identify with the space of "characteristic polynomials" $\mathfrak{g} \rightarrow \mathfrak{c}_{\mathfrak{g}^*}$. The fiber in $\mathfrak{g}^{\text{rss}}$ over each element of $\mathfrak{c}_{\mathfrak{g}^*}$ is the definable stable conjugacy class with the given characteristic polynomial. Note that $\mathfrak{c}_{\mathfrak{g}^*}$ depends only on the quasi-split form G^* . Similarly, we have a map $G \rightarrow \mathfrak{c}_{G^*} := (T^{**}/W)^*$, where T^{**} is a maximally split Cartan subgroup of the split form G^{**} of G , and W is the absolute Weyl group for T^{**} in G^* . By \mathfrak{c}_{G^*} we mean the twisted form of T^{**}/W obtained from the outer automorphisms of T^{**} used to define G^* . That is, the Galois group $\text{Gal}(K/\text{VF})$ acts so that VF-points are

$$t = \phi_\sigma(\sigma(t)) \mod W, \quad \forall \sigma \in \text{Gal}(K/\text{VF}),$$

where $t \mapsto \sigma(t)$ is standard action of the Galois group on $T^{**}(K)$, and ϕ_σ is the outer automorphism associated with σ .

If F is a nonarchimedean field, and $z \in Z(F)$, we have the fiber G_z . It is a connected reductive linear algebraic group with Lie algebra \mathfrak{g}_z . The map $\mathfrak{g}_z \rightarrow \mathfrak{c}_{\mathfrak{g}^*, z}$ is the usual map from the Lie algebra to the space of characteristic polynomials.

2.2.3. *Endoscopic data.* We may similarly use the endoscopic data from our fixed choices to form a space of quasi-split endoscopic data for H , again split by K/VF . We let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H .

We similarly construct a cocycle space for H and take the fiber product Z of the cocycle spaces for G and H , again denoting it by Z , by abuse of notation. We also include in the parameters of the cocycle space Z a large unramified extension and a choice qFr of generator. We also include in Z a parameter b that trivializes an invariant differential form of top degree, as explained in Section 2.3.

2.2.4. *Nilpotent elements.* The properties of nilpotent elements are well-known. Here we summarize the properties that we use.

Theorem 7. *Let $G_{/Z}$ be a definable reductive group with Lie algebra $\mathfrak{g}_{/Z}$ over a cocycle space Z . There exists $m \in \mathbb{N}$ such that for all $F \in \text{Loc}_m$, and all $z \in Z(F)$, the following properties are equivalent properties of $N \in \mathfrak{g}_z(F)$:*

- (1) $\rho(N)$ is nilpotent for some faithful rational representation ρ of \mathfrak{g}_z ;
- (2) $\rho(N)$ is nilpotent for every faithful rational representation ρ of \mathfrak{g}_z ;
- (3) 0 lies in the Zariski closure of the adjoint orbit of N ;
- (4) 0 lies in the p -adic closure of the adjoint orbit of N ;
- (5) there exists $\lambda \in X_*^F(G_z)$ such that $\lim_{t \rightarrow 0} \text{Ad}(\lambda(t))N = 0$; and
- (6) the image of N under the morphism $\mathfrak{g}_z \mapsto \mathfrak{c}_{\mathfrak{g}_z^*}$ is 0 .

Proof. All the implications mentioned in this proof should be understood as implications for the residual characteristic sufficiently large, with effective bound that depends only on the absolute root datum of G . Such bounds are stated in the sources we cite. The equivalence of the first two properties is in [19, §15.3]. The implications (5) \Leftrightarrow (4) \Rightarrow (3) are in [14], [3, 2.5.1]. The implications (2) \Leftrightarrow (3) \Rightarrow (5) are in [27, 3.5], [27, 4.1, Prop. 4 and Theorem 26].

To see the equivalence of (6) with say (2), we may work over an algebraically closed field. In particular, $\mathfrak{g} = \mathfrak{g}^{**}$. Note that the image of N is 0 in $\mathfrak{c}_{\mathfrak{g}}$ if and only if the image of N is 0 in the torus $\mathfrak{b}/\mathfrak{n}$ (under conjugation to a Borel subalgebra), which holds if and only if the image of N is in \mathfrak{n} , the nilradical of \mathfrak{b} . This holds if and only if the semisimple part of N is trivial, which is equivalent to (2). \square

The last of the properties enumerated in the theorem is a definable condition. Thus, we have a definable set of all nilpotent elements in \mathfrak{g} . The theorem gives the compatibility of this definition with notions of nilpotence in the various articles we cite. The following theorem appears in [5].

Theorem 8 (Barbasch-Moy). *Let $G_{/Z}$ and $\mathfrak{g}_{/Z}$ be as above. There exists m such that for all $F \in \text{Loc}_m$, $z \in Z(F)$, and for every nilpotent orbit in $\mathfrak{g}_z(F)$, there exists $\mathfrak{f} \in \mathcal{F}$ and a nilpotent element N in the orbit such that*

- (1) $N \in \mathfrak{g}_{z,\mathfrak{f},0}$, and
- (2) if N' is nilpotent and $N' \in N + \mathfrak{g}_{z,\mathfrak{f},0+}$, then N lies in the p -adic closure of the orbit of N' .

We say that $\Upsilon = (N, \mathfrak{f})$ is a Barbasch-Moy pair, with $\mathfrak{f} \in \mathcal{F}$, if N is nilpotent and it satisfies the properties of the theorem. For each $\mathfrak{f} \in \mathcal{F}$, there is a definable set consisting of all nilpotent elements N such that $\Upsilon = (N, \mathfrak{f})$ is a Barbasch-Moy pair.

Corollary 9. *Let $G_{/Z}$ and $\mathfrak{g}_{/Z}$ be as above. There exists a constant k such that for all $F \in \text{Loc}_m$, $z \in Z(F)$, the number of nilpotent conjugacy classes in $\mathfrak{g}_z(F)$ is at most k .*

Proof. By the preceding theorem, for given F , the number of nilpotent classes is at most the sum of the numbers $k_{\mathfrak{f}}$, where $k_{\mathfrak{f}}$ is the number of nilpotent conjugacy classes in the finite reductive group $\mathfrak{g}_{z,\mathfrak{f},0}/\mathfrak{g}_{z,\mathfrak{f},0+}$. Field-independent bounds on the number of nilpotent elements in a reductive group over a finite field are well-known [6]. \square

We have corresponding versions of these theorems for the set of unipotent elements. We have a definable subset of G consisting of unipotent elements u , determined by the condition that the image of u under $G \rightarrow \mathfrak{c}_{G^*}$ is 1.

Theorem 10. *Let $G_{/Z}$ be as above. There exists m such that for all $F \in \text{Loc}_m$, $z \in Z(F)$, and for all unipotent conjugacy classes in $G_z(F)$, there exists $\mathfrak{f} \in \mathcal{F}$ and an element u in the unipotent conjugacy class such that*

- (1) $u \in G_{z,\mathfrak{f},0}$, and
- (2) if u' is unipotent and $u' \in uG_{z,\mathfrak{f},0+}$, then u lies in the p -adic closure of the conjugacy class of u' .

Proof. First, we show that (1) and (2) are first-order statements in the Denef-Pas language. The statement (2) is first-order because the set $G_{z,\mathfrak{f},0+}$ is definable, by [9, Lemma 3.4]. The set $G_{z,\mathfrak{f},0}$ appearing in (1) is not currently known to be definable in general, but by the same lemma, it is contained in a slightly larger definable set $G_{\mathfrak{f}}^0$ of Galois-fixed points of $G_z(K)_{\mathfrak{f},0}$ (where K is an extension that splits G_z). However, if u is a unipotent element contained in $G_{\mathfrak{f}}^0$, then in fact u is contained in $G_{z,\mathfrak{f},0}$ by [14, Lemma 4.5.1]. Thus (2) is a first-order statement as well.

Because these statements are first order in the Denef-Pas language, by a transfer principle, it is enough to prove the result when F has characteristic zero. Now we work over F , with $z \in Z(F)$ fixed, and write $\mathfrak{g}_{\mathfrak{f},r}$, $\mathfrak{g}_{\mathfrak{f},r+}$, $G_{\mathfrak{f},r}$, $G_{\mathfrak{f},r+}$ for the usual Moy-Prasad filtrations with $r \in \mathbb{R}$ for the Lie algebra, and $r \in \mathbb{R}_{\geq 0}$ for the group. In particular, $\mathfrak{g}_{\mathfrak{f},0} = \mathfrak{g}_{z,\mathfrak{f},0}(F)$, and so forth.

When F is characteristic zero, and m is large enough, we have an exponential map defined on the set of all topologically nilpotent elements with the following properties.

- (1) $\exp(\mathfrak{g}_{\mathfrak{f},r}) = G_{\mathfrak{f},r}$, for all $r > 0$.
- (2) $\exp(\mathfrak{g}_{\mathfrak{f},0+}) = G_{\mathfrak{f},0+}$.
- (3) \exp restricts to a bijection between the set of nilpotent elements in $\mathfrak{g}_{\mathfrak{f},0}$ and the set of unipotent elements in $G_{\mathfrak{f},0}$.
- (4) Let N be a nilpotent element in $\mathfrak{g}_{\mathfrak{f},0}$. Then the image of $\exp(N)$ in $G_{\mathfrak{f},0}/G_{\mathfrak{f},0+}$ is $\text{fexp}(\bar{N})$, where \bar{N} is the image of N in $\mathfrak{g}_{\mathfrak{f},0}/\mathfrak{g}_{\mathfrak{f},0+}$ and fexp is the finite field exponential from the nilpotent set of $\mathfrak{g}_{\mathfrak{f},0}/\mathfrak{g}_{\mathfrak{f},0+}$ to the unipotent set of $G_{\mathfrak{f},0}/G_{\mathfrak{f},0+}$. The map fexp is injective on the nilpotent set.
- (5) The exponential map from the nilpotent set to the unipotent set preserves orbits and the partial order given by orbit closure.

These properties follow from [21, Prop. 3.1.1], which shows that the exponential map defined by the usual series converges, and from the properties of mock exponential maps constructed by Adler [1, §1.5-1.6]. When the exponential is defined, it is a mock exponential map.

Pick u in the given conjugacy class in such a way that $u = \exp(N)$, and N has the properties in Theorem 8, for some $\mathfrak{f} \in \mathcal{F}$. Then $u \in G_{z,\mathfrak{f},0}$. Let $u' = \exp(N') \in uG_{z,\mathfrak{f},0+}$. Reducing modulo $G_{\mathfrak{f},0+}$ gives $\text{fexp}(\bar{N}') = \text{fexp}(\bar{N})$. By injectivity, we have $\bar{N}' = \bar{N}$ and $N' \in N + \mathfrak{g}_{\mathfrak{f},0+}$. By Theorem 8, N lies in the closure of the orbit of N' . Exponentiating again, $u = \exp(N)$ lies in the closure of the orbit of $u' = \exp(N')$. \square

We say that $\Upsilon = (u, \mathfrak{f})$ is a unipotent Barbasch-Moy pair, with $\mathfrak{f} \in \mathcal{F}$, if u is unipotent and it satisfies the properties of the theorem. For each $\mathfrak{f} \in \mathcal{F}$, there is a definable set consisting of all unipotent elements u such that $\Upsilon = (u, \mathfrak{f})$ is a unipotent Barbasch-Moy pair.

2.3. Volume forms. All integrals are to be computed with respect to invariant measures on their respective orbits. This means that the appropriate context for motivic integration is integration with respect to volume forms, as described in [12, §8]. Volume forms in the context of reductive groups are explained in [10]. We follow those sources.

We recall that each nilpotent orbit can be endowed with an invariant motivic measure $d\mu^{\text{nil}}$ that comes from the canonical symplectic form on coadjoint orbits [10, Prop. 4.3].

We obtain an invariant volume form on the unipotent set in the group by exponentiating the volume form on the nilpotent cone.

The invariant volume forms on stable regular semisimple orbits in the Lie algebra are constructed in [11]. We have the morphism $\mathfrak{g}^{\text{rss}} \rightarrow \mathfrak{c}_{\mathfrak{g}^*}$ that classifies the regular semisimple stable orbits. The Leray residue (in the sense of [12]) of the canonical volume form on \mathfrak{g} by the canonical volume form on $\mathfrak{c}_{\mathfrak{g}^*}$ yields an invariant volume form $d\mu^{\text{rss}}$ on each fiber of the morphism, and hence an invariant volume form on stable regular semisimple orbits with free parameter $X \in \mathfrak{c}_{\mathfrak{g}^*}$. We compute all stable orbital integrals on stable regular semisimple orbits with respect to this family of volume forms. By restriction to each conjugacy class in the stable conjugacy class, we obtain an invariant measure on each regular semisimple conjugacy class.

We construct an algebraic differential d -form on G , where d is the relative dimension of G over Z . We begin with the split case (and Z a singleton set). If G^{**} is split over VF, we have a top invariant form on G^{**} given on an open cell of G^{**} by

$$(3) \quad d^{\times}t \wedge dn \wedge dn',$$

where $d^{\times}t$ is a top invariant form on a split torus $T^{**} \subset B^{**}$, dn a top invariant form on the unipotent radical $R_u(B)^{**}$ of B^{**} , and dn' on the unipotent radical of the Borel subgroup opposite to B^{**} along T^{**} . We may pick root vectors X_{α} for each root (as fixed choices over \mathbb{Q}). Then by placing a total order on the positive roots, we may write

$$n = \prod_{\alpha > 0} \exp(x_{\alpha} X_{\alpha}),$$

and $dn = \wedge_{\alpha > 0} dx_{\alpha}$. Similar considerations hold for dn' . It is known that this differential form extends to an invariant differential form on all of G^{**} .

To go from the split case to the general case, let G/Z be a definable reductive group over a cocycle space Z . Part of the data for Z is a field extension K splitting G . Let G_K be the (definable) base change of G from VF to K . We may view G as a subset of $G_K \subset \text{VF}^n$, for some $n \in \mathbb{N}$.

We may extend the top invariant form ω to ω_K , from VF to $K = \text{VF}[t]/(p_a(t)) = \text{VF}^k$, by taking each coordinate x_{α} , and so forth to be a monic polynomial of degree k with coefficients in VF. This does not change the degree d of the differential form, but expanding in t , it takes values in VF^k . The form ω_K is G_K -invariant. It is also invariant by the action of $G_{K, \text{adj}}$ on G_K by conjugation. We have a morphism $G_K \rightarrow Z$, and G is identified with the fixed point set of the action of $\text{Gal}(K/\text{VF})$ on G_K . Let dz be any top form on Z .

We describe the differential form when G is quasi-split. In this case, the action of the Galois group fixes a pinning, the Galois group permutes the root spaces and permutes the root vectors X_α , up to some structure constants. From the expression for the differential form in Formula 3, we see that the Galois group preserves the differential form $dz \wedge \omega_K$ up to a cocycle with values in K^\times .

Even if the group is not quasi-split, the Galois group fixes the pinning up to a cocycle g_σ with values in $G_{K,\text{adj}}$, and because the adjoint group action preserves $dz \wedge \omega_K$, we see that the differential form is preserved by the Galois action, up to a cocycle a_σ with values in K^\times . When we take nonarchimedean structures, we may split this cocycle by Hilbert's 90th. Working at the definable level, we introduce a new free parameter b into the cocycle space Z taking values in K^\times that is subject to the relation $a_\sigma = \sigma(b)^{-1}b$.

Set $\omega_0 = b dz \wedge \omega_K$. It is Galois stable on G , so on this definable set, the $d + \dim Z$ -form takes values in F . Also, since ω_K is invariant by G_K , this implies the invariance of ω_0 by G (acting fiberwise over Z).

We have a morphism $G^{\text{rss}} \rightarrow \mathfrak{c}_{G^*}$ that classifies stable regular semisimple conjugacy classes in the group. We form the Leray residue of the Haar volume form ω_0 on G with respect to the canonical volume form on \mathfrak{c}_{G^*} . This yields an invariant volume form on each fiber of the morphism, and hence an invariant volume form $d\mu^{\text{rss}}$ on stable regular semisimple orbits in the group.

2.4. Relation to valued fields. To put the preceding definitions in context, we make a few comments about reductive groups over nonarchimedean fields F in large residual characteristic. In applications, we assume that the residual characteristic is sufficiently large that all Galois groups that appear are tame (that is, wild inertia is trivial).

2.4.1. Multiplicative characters. We recall that there is currently no good theory of multiplicative characters for motivic integration. In particular, since the Langlands-Shelstad transfer factor makes use of multiplicative characters in the form of χ -data, when working motivically, we restrict our attention to a neighborhood of the identity on the group. That is, we restrict to the kernels of the multiplicative characters.

2.4.2. Frobenius automorphism. Although we may speak of unramified extensions and their Galois groups, when working motivically, we have no way to single out the canonical Frobenius generator of the Galois group of an unramified extension. Indeed, we do not have access the cardinality of the residue field q , until we specialize to a particular nonarchimedean field F .

Instead, in [11] and here, we work with an arbitrary generator qFr of cyclic group Σ^{unr} . In the Tate-Nakayama isomorphism, the quasi-Frobenius element is used to identify $\text{Gal}(E/F) = \mathbb{Z}/n\mathbb{Z}$, when E/F is unramified of degree n . If we pick the “wrong” (i.e. non-Frobenius) generator of $\text{Gal}(E/F)$, it has the effect of replacing the endoscopic datum $s \in \hat{T}^\Sigma$ with another element s^i with the same centralizer, for some $i \in (\mathbb{Z}/n\mathbb{Z})^\times$. This again gives valid endoscopic data, and the arguments still work.

3. LANGLANDS-SHELSTAD TRANSFER FACTOR FOR LIE ALGEBRAS

In this section, we prove the constructibility of the Langlands-Shelstad transfer factor for Lie algebras as stated in Theorem 3. Our definition will be carried out entirely in terms of

the Denef-Pas language and constructible functions, without reference to a nonarchimedean field. We note that this theorem has already been established in the unramified case [11]. All the ideas of the proof are already present in the unramified case. The key observation is that the transfer factor can be built up from Tate-Nakayama, field extensions, cocycles, $SL(2)$ -triples, conjugation, group operations, and inner twisting. These concepts have all been developed in a definable context by previous research [11], [10]. Readers who are willing to accept without proof the constructibility of transfer factors may skip ahead to Section 4.

We make a fixed choice of a nonzero sufficiently divisible $k \in \mathbb{Z}$. We will give the definition of the transfer factor in backwards order starting with the top-level description, and successively expanding and refining the definitions until all unknown terms have been specified. The transfer factor $\Delta(X_H, X_G, \bar{X}_H, \bar{X}_G)$ is the product of two constructible terms:

$$\Delta_0(X_H, X_G, \bar{X}_H, \bar{X}_G) \quad \text{and} \quad \mathbb{L}^{d(X_H, X_G)}.$$

The parameters X_H and \bar{X}_H run over the definable set of G -regular semisimple elements of the Lie algebra \mathfrak{h} . The parameters X_G and \bar{X}_G run over the definable set of regular semisimple elements of \mathfrak{g} ; that is, $(X_G, \bar{X}_G) \in \mathfrak{g}^{\text{rss}} \times_Z \mathfrak{g}^{\text{rss}}$. Recall that ψ is the inner isomorphism defined over an extension of VF between G and G^* . The transfer factor is defined to be 0 unless X_H and X_G correspond under the definable condition requiring the image of $\psi(X_G)$ in the Chevalley quotient $\mathfrak{c}_{\mathfrak{g}^*}$ to equal the image of X_H in $\mathfrak{c}_{\mathfrak{h}}$ under $\mathfrak{c}_{\mathfrak{h}} \rightarrow \mathfrak{c}_{\mathfrak{g}^*}$, and similarly for (\bar{X}_H, \bar{X}_G) . (Note that the endoscopic groups H is quasi-split by definition, so that $H = H^*$ and $\mathfrak{h} = \mathfrak{h}^*$.) We now assume that the parameters are restricted to the definable subsets satisfying these constraints. The factor $\mathbb{L}^{d(X_H, X_G)}$ is the usual discriminant factor (called Δ_{IV} by Langlands and Shelstad). It is constructible by [11].

The parameter X_G is to be considered the primary parameter. The transfer factor depends in a subtle way on its conjugacy class within its stable conjugacy class. The parameters (\bar{X}_H, \bar{X}_G) should be viewed as secondary, only affecting the normalization of the transfer factor by a scalar independent of X_G .

The constructible function Δ_0 is a step function on

$$(4) \quad D := \mathfrak{h}^{G-\text{rss}} \times_Z \mathfrak{g}^{\text{rss}} \times_Z \mathfrak{h}^{G-\text{rss}} \times_Z \mathfrak{g}^{\text{rss}}$$

with variables $(X_H, X_G, \bar{X}_H, \bar{X}_G)$, and it will be defined as a finite combination of indicator functions $1_{D_{j,*}}$ of definable sets $D_{j,\text{I}}$ and $D_{j,\text{III}}$:

$$\Delta_0 = \left(\sum_{j_1=0}^{k-1} e^{2\pi i j_1/k} 1_{D_{j_1,\text{III}}} \right) \left(\sum_{j_2=0}^{k-1} e^{2\pi i j_2/k} 1_{D_{j_2,\text{I}}} \right) \left(\sum_{j_3=0}^{k-1} e^{-2\pi i j_3/k} 1_{D_{j_3,\text{I}}} \right).$$

The subscripts correspond to the Roman numerals I and III₁ in the Langlands-Shelstad article. Typically, constructible functions have integer coefficients, but there is no harm in extending scalars to \mathbb{C} to allow the given roots of unity.

We will give further definable sets $D_{j,\text{I}}$ and $D_{j,\text{III}}$. The definable set $D_{j,\text{III}}$ is a subset of the D . The definable set $D_{j,\text{I}}$ is a subset of $\mathfrak{h}^{G-\text{rss}} \times_Z \mathfrak{g}^{\text{rss}}$, which we pull back to sets $D_{j,\text{I}}^{12}$ and $D_{j,\text{I}}^{34}$ on D under the projections maps onto the first two factors, and the last two factors of D , respectively:

$$(X_H, X_G, \bar{X}_H, \bar{X}_G) \mapsto^{12} (X_H, X_G), \quad (X_H, X_G, \bar{X}_H, \bar{X}_G) \mapsto^{34} (\bar{X}_H, \bar{X}_G).$$

We have a space of regular nilpotent elements in the quasi-split Lie algebra \mathfrak{g}^* . The Kostant section [22] to the Chevalley quotient is determined by a choice of regular nilpotent element. The transfer factor will be independent of the choice of regular nilpotent used to determine the Kostant section, allowing us to treat the regular nilpotent element as a bound parameter. The element $X_H \in \mathfrak{h}$ determines an element $X \in \mathfrak{g}^*$ by the composite of maps

$$\mathfrak{h} \rightarrow \mathfrak{c}_{\mathfrak{h}} \rightarrow \mathfrak{c}_{\mathfrak{g}^*} \rightarrow \mathfrak{g}^*,$$

where the last map is the Kostant section.

In the rest of this section, we review the technical details of the construction from [26]. These details require a significant amount of local notation ($\lambda(T_{\text{sc}})$, s_{α} , $n(s_{\alpha})$, $m(\sigma_T)$, a_{α} , ω_T , p , inv , h , \bar{h} , U , $u(\sigma)$, $v(\sigma)$, $\bar{v}(\sigma)$, κ_T , κ_U) that has been borrowed from [26]. This notation will not be used elsewhere in this article.

We define a -data to be the collection of constants $a_{\alpha} := \alpha(X)$, for $\alpha \in \Phi$. This choice of a -data allows us to do without χ -data (in the sense of Langlands and Shelstad) for the Lie algebra transfer factor.

3.1. The cocycle $\lambda(T_{\text{sc}})$. We recall the definition of a Galois cocycle $\lambda(T_{\text{sc}})$ with values in $T_{\text{sc}}(K)$, attached to the centralizer T_{sc} of X in G_{sc}^* , where G_{sc}^* is the simply connected cover of the derived group of G^* . The entire construction takes place in the simply connected cover. In this one subsection, we sometimes drop the subscript sc , writing (G^*, B^*, T^*) instead of $(G_{\text{sc}}^*, B_{\text{sc}}^*, T_{\text{sc}}^*)$, and so forth.

We take K/VF to be a splitting field of G , and take L/VF to be a Galois extension that splits T_{sc} . There are bound parameters running over L/VF and its enumerated Galois group. We let h be a bound parameter in G^* such that $(T_{\text{sc}}, B_{\text{sc}}) = \text{Ad}(h)(T^*, B^*)$.

If w is any element of the Weyl group of T^* in G^* with reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_r}$, we set $n(w) = n(s_{\alpha_1}) \cdots n(s_{\alpha_r})$ where $n(s_{\alpha})$ is the image in $\text{Norm}_{G_{\text{sc}}^*}(T_{\text{sc}}^*)$ of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp X_{\alpha} \exp -X_{-\alpha} \exp X_{\alpha}$$

under the homomorphism $SL(2) \rightarrow G_{\text{sc}}^*$ attached to the Lie triple $\{X_{\alpha}, X_{-\alpha}, H_{\alpha}\}$ coming from a pinning $(B^*, T^*, \{X_{\alpha}\})$. The transfer factor does not depend on the pinning, and we treat the pinning as a bound parameter. The exponential map used to define $n(s_{\alpha})$ is a polynomial function on nilpotent elements.

The cocycle $\lambda(T_{\text{sc}})$ is defined using the a -data by an enumerated cocycle of $\text{Gal}(K/\text{VF})$ as described above, with

$$\sigma \mapsto hm(\sigma_T)\sigma(h)^{-1}, \quad \text{where} \quad m(\sigma_T) := \left(\prod_{1, \sigma}^p a_{\alpha}^{\alpha^{\vee}} \right) n(\omega_T(\sigma)),$$

and where $x^{\alpha^{\vee}}$ denotes the image of x under the coroot α^{\vee} . An inspection of each element of the formula shows that this cocycle is definable. See [26] for the combinatorial description of the gauge p and the choice ω_T of Weyl group elements (coming from the twisted action of the Galois group on T_{sc}^*).

3.2. Another cocycle. Next we recall the definition of an enumerated Galois cocycle $\text{inv}(X_H, X_G, \bar{X}_H, \bar{X}_G)$. Let T and \bar{T} be the centralizers in G of X and \bar{X} , where X and \bar{X} are constructed as above from a Kostant section. We take a bound parameter space for a Galois field extension $L/K/\text{VF}$ that splits T and \bar{T} . The cocycle takes values in the L -points of $U := (T \times \bar{T})/\text{Zent}_{\text{sc}}$, where the center Zent_{sc} of the simply connected cover is mapped diagonally.

To construct the cocycle, we take bound parameters h and \bar{h} in $G(L)$ such that

$$\text{Ad } h \psi(X_G) = X, \quad \text{Ad } \bar{h} \psi(\bar{X}_G) = \bar{X},$$

with ψ the inner twist given above. As an inner twist, we get an enumerated cocycle of $\text{Gal}(K/\text{VF})$ by $\text{Int } u(\sigma) := \psi\sigma(\psi)^{-1}$. By our embedding $K \rightarrow L$ we obtain an enumerated cocycle of $\text{Gal}(L/\text{VF})$. Set $v(\sigma) = hu(\sigma)\sigma(h)^{-1}$ and $\bar{v}(\sigma) = \bar{h}u(\sigma)\sigma(\bar{h})^{-1}$. The cocycle $\text{inv}(X_H, X_G, \bar{X}_H, \bar{X}_G)$ is then

$$\sigma \mapsto (v(\sigma)^{-1}, \bar{v}(\sigma)) \in U(L).$$

All this data is definable.

3.3. Tate-Nakayama. We continue with our fixed choice of a sufficiently divisible integer k . We have an element $\kappa \in \hat{T}$ that is part of our endoscopic data. The article [11] shows how to treat this element as an enumerated cocycle κ_T with values in $X^*(T)$ for various Cartan subgroups T . It uses this to give the Tate-Nakayama pairing in terms of the Denef-Pas language. (This requires the cocycle space of Z to include the space of pairs (E, qFr) of an unramified extension of sufficiently large fixed degree and generator qFr of the Galois group E/VF .) In particular, with the fixed choice of a highly divisible integer k as above, for $j \in \mathbb{Z}$, and for a definable parameter space of enumerated cocycles $c(\sigma)$ with values in $T(L)$, there is a definable set

$$\text{TN}(c(\sigma), \kappa_T)_j$$

of all cocycles such that the Tate-Nakayama pairing of the cocycle with κ_T has value $e^{2\pi j/k}$. We define $D_{j,\text{I}}(X_H, X_G)$ to be

$$\text{TN}(\lambda(T_{\text{sc}}), \kappa_{T_{\text{sc}}})_j$$

Similarly, we may form an element κ_U (the image of $(\kappa_T, \kappa_{\bar{T}})$ in $X^*(U)$). We let

$$D_{j,\text{III}}(X_H, X_G, \bar{X}_H, \bar{X}_G)$$

be the definable set

$$\text{TN}(\text{inv}(X_H, X_G, \bar{X}_H, \bar{X}_G), \kappa_U)_j$$

Combined with our earlier definitions, this completes the definition of the Lie algebra Langlands-Shelstad transfer factor as a constructible function.

Each part of the definition is a direct translation of the definition over nonarchimedean fields, adapted to the Lie algebra. The choice $a_\alpha = \alpha(X)$ causes the term Δ_{Π} in the Langlands-Shelstad definition to equal 1. For the Lie algebra transfer factor, we may take $\Delta_2 = 1$.

4. TRANSFER PRINCIPLES

4.1. Transfer of asymptotic identities. Our aim is to transfer identities of Shalika germs from one field to another. Since germs express the asymptotics of orbital integrals, we develop a transfer principle for asymptotic identities. We have the following transfer principle.

Theorem 11. *Let S be a definable set. Let g be a constructible function on S and let $f : S \rightarrow \mathbb{Z}$ be a definable function. For each $a \in \mathbb{Z}$, let $S_a = f^{-1}(a)$. Then there exists $m \in \mathbb{N}$ such that for all $F_1, F_2 \in \text{Loc}_m$ with the same residue field, we have the following: for all $a \in \mathbb{Z}$, g_{F_1} is zero on $S_a(F_1)$ if and only if g_{F_2} is zero on $S_a(F_2)$.*

Proof. We use the notation of the cell decomposition theorem [12, Theorem 7.2.1]. In particular, in this one proof, Z will denote a cell, in a departure from its usual meaning as a cocycle space. By quantifier elimination, we may assume that none of our formulas contain bound variables of the valued field sort. Assume that $S \subset S'[1, 0, 0]$, for some definable set S' . By cell decomposition, we may partition S into finitely many cells Z such that each cell comes with a definable isomorphism $\iota : Z \rightarrow Z' \subset S'[1, n', n'']$ and a projection π of the cell Z' onto a base $C \subset S'[0, n', n'']$. Furthermore, there exists a constructible function g_C on each base C such that

$$g|_Z = p^* g_C, \quad \text{where } p = \pi \circ \iota.$$

The morphism p is an isomorphism followed by a projection of a cell to its base. As such, there exists m such that for all $F \in \text{Loc}_m$, the map p_F is onto.

Similarly, we may pick a cell decomposition of S adapted to the definable function $f : S \rightarrow \mathbb{Z}$. (We use a slightly stronger property than what is stated in [12]. Since f is definable, f_C is also definable, rather than merely constructible.) By [12, Prop 7.3.2], there is a common cell refinement that is adapted to both g and f .

Note that the base C has one fewer valued field variable than S . We iterate this construction, to eliminate all valued field variables. After iteration, and choosing new constants m , n' , and n'' , we have the following situation. S can be partitioned into finitely many definable sets Z , each equipped with a morphism $p : Z \rightarrow C \subset h[0, n', n'']$. Furthermore, for each C there is a definable function $f_C : C \rightarrow \mathbb{Z}$ and constructible function g_C on C such that

$$f|_Z = p^* f_C, \quad g|_Z = p^* g_C.$$

The morphisms $p_F : Z(F) \rightarrow C(F)$ are onto, for $F \in \text{Loc}_m$.

Let $F_1, F_2 \in \text{Loc}_m$ have the same residue field. Since g_C and f_C have no valued field variables, we may assume that $f_{C, F_1} = f_{C, F_2}$ and $g_{C, F_1} = g_{C, F_2}$.

The set S_a is definable. Assume that $g|_{S_a, F_1} = 0$. Then on any part Z ,

$$0 = g|_{S_a \cap Z, F_1} = p_{F_1}^* g_{C_a, F_1}, \quad \text{where } C_a = f_C^{-1}(a)$$

Since p_{F_1} is onto, g_{C_a, F_1} is identically zero, and thus so is g_{C_a, F_2} and $0 = p_{F_2}^* g_{C_a, F_2} = g|_{S_a \cap Z, F_2}$. Thus, $g|_{S_a, F_2} = 0$. This proves the theorem. \square

Corollary 12. *Let g be a constructible function on $S \times \mathbb{Z}$. There exists m such for all $F_1, F_2 \in \text{Loc}_m$ with the same residue field, we have the following. If for some $a_0 \in \mathbb{Z}$, the function $g(\cdot, a)_{F_1}$ is identically zero on $S(F_1)$ for all $a \geq a_0$, then the function $g(\cdot, a)_{F_2}$ is also identically zero on $S(F_2)$ for all $a \geq a_0$.*

Proof. Let $f : S \times \mathbb{Z} \rightarrow \mathbb{Z}$ be the projection onto the second factor. The preimage of a under f is $S \times \{a\}$. Apply the theorem to g and f for each $a \geq a_0$. \square

We may apply the theorem and corollary to obtain a transfer principle for asymptotic relations as follows. Let g be a constructible function on S and let 1_a be a definable family of functions on S indexed by $a \in \mathbb{Z}$. For example, 1_a might be a family of characteristic functions of a shrinking family of neighborhoods of a point s_0 . (In this article, we often use the notation 1_X quite loosely to denote families of indicator functions *indexed* by some X running over a definable set. We do not require 1_X to be the indicator function of X itself.) Then we obtain a constructible function on $S \times \mathbb{Z}$ by $(s, a) \mapsto g(s)1_a(s)$. The corollary applied to this constructible function gives a transfer principle for the vanishing of g in sufficiently small neighborhoods of the point s_0 . The size a_F of the neighborhood is allowed to vary with the field F .

4.2. Uniformity of asymptotic relations. We use the following lemma based on [10, Th 4.4.4].

Lemma 13. *Let $\Lambda \times S$ be a definable set. Let g be a constructible function on $\Lambda \times S$. Then there exists $m \in \mathbb{Z}$ and a constructible function $\text{Iv}(g)$ on Λ , such that for all $F \in \text{Loc}_m$, the zero locus of $\text{Iv}(g)_F$ equals the locus of*

$$\{v \in \Lambda(F) \mid \forall s \in S(F), \quad g_F(a, s) = 0\}.$$

Proof. Theorem 4.4.4 of [10] works with exponential constructible functions instead of constructible functions, but the identical proof applies when working with non-exponential functions.

Also, that theorem is restricted to $S = h[n, 0, 0]$. We show that this special case implies the more general statement of our lemma. First, the case $h[n, n', n'']$ reduces to $h[n + n' + n'', 0, 0]$ by replacing each \mathbb{Z} -variable with $\text{ord } x$ for some new VF parameter x , and replacing each residue field variable with some $\overline{\text{ac}} x$ for some new VF parameter x .

If $S \subset h[n, n', n'']$ is arbitrary, then we replace the function g with the function $g1_S$, where 1_S is the indicator function on S to go from $h[n, n', n'']$ to S . We are now in the context covered by Theorem 4.4.4. \square

We can show that the bound a_0 in the Corollary 12 can be chosen uniformly in the following sense.

Theorem 14. *Let g be a constructible function on a definable set $S \times \mathbb{Z}$. Suppose that there exists $m \in \mathbb{Z}$ such that for all $F \in \text{Loc}_m$, there exists a_F for which $g(\cdot, a)_F$ is identically zero on $S(F)$, for all $a \geq a_F$. Then there exists $m' \geq m$ and a single a_0 such that we can take $a_F = a_0$ for all $F \in \text{Loc}_{m'}$.*

Proof. We apply the lemma with $\Lambda = \mathbb{Z}$ to obtain a constructible function $\text{Iv}(g)$ on \mathbb{Z} that describes the locus of identical vanishing of g , which is a subset of \mathbb{Z} depending on $F \in \text{Loc}_m$.

A constructible function $\text{Iv}(g)$ lies in the tensor product $Q(\mathbb{Z}) \otimes_{P^0(\mathbb{Z})} P(\mathbb{Z})$. The exact definitions of Q and P^0 are not important to us. See [8]. Nonetheless, it is important that this tensor product produces a separation of variables for $\text{Iv}(g) = \sum q_i \otimes p_i$, where all residue field variables occur in the ring $Q(\mathbb{Z})$ on the left, and where the ring $P(\mathbb{Z})$ on the right is the ring of constructible Presburger functions on \mathbb{Z} . We may assume by cell decomposition that $\text{Iv}(g)$ contains no variables of valued field sort.

By the definition of constructible Presburger functions p_i on \mathbb{Z} , we may partition \mathbb{Z} into a finite (field independent) disjoint union of Presburger sets such that on each of these sets, $\text{Iv}(g)$ has the form

$$\text{Iv}(g)_F(t) = \sum_{i=1}^n c_i t^{k_i} q_F^{\ell_i t},$$

where q_F is the cardinality of the residue field of F , each c_i depends only on residue field variables, the integers k_i and ℓ_i do not depend on any variables in the Denef-Pas language, and we can assume that (k_i, ℓ_i) are mutually different for different i .

The key point is that such a function can have only finitely many zeroes, and their number is bounded by a constant that depends only on the number n of terms in the sum. This is [8, Lemma 2.1.7] and follows from o -minimality of the structure of the real field enriched with the exponential. Thus, the only way for the zero locus of $\text{Iv}(g)$ to contain all integers greater than a_F is for the coefficients c_i to be zero for all fields F for every unbounded Presburger set in the disjoint union.

If we now take a_0 to be larger than the maximum of all the bounded Presburger sets in the the disjoint union, then this integer works uniformly for all fields F . \square

4.3. Transfer of linear dependence. We use the following result from [9]. We conjectured this result to hold based on the requirements of smooth endoscopic matching.

Theorem 15 (Cluckers-Gordon-Halupczok). *Let $g : P \rightarrow \Lambda$ be a definable morphism between definable sets. For any constructible function f on P , write f_λ for the constructible function on the fiber P_λ over $\lambda \in \Lambda$. Let \mathbf{f} be a tuple of constructible functions on P . For $\lambda \in \Lambda$, let \mathbf{f}_λ be the corresponding tuple of constructible functions on the fiber P_λ . Then there exists a natural number m with the following property. For every $F_1, F_2 \in \text{Loc}_m$ such that F_1 and F_2 have isomorphic residue fields, the following holds: if for each $\lambda \in \Lambda(F_1)$, the tuple of functions $\mathbf{f}_{\lambda, F_1}$ is linearly dependent, then also for each $\lambda_2 \in \Lambda(F_2)$, the tuple of functions $\mathbf{f}_{\lambda_2, F_2}$ is linearly dependent.*

Let $S' \subset S \times \mathbb{Z}$ be any sets. (This particular definition pertains to sets of set theory rather than definable sets.) For $a_0 \in \mathbb{Z}$, we say that a tuple of complex-valued functions (f_1, \dots, f_r) is a_0 -asymptotically linearly dependent if for all $a \geq a_0$, the tuple of functions $(f_1|_{S_a}, \dots, f_r|_{S_a})$ is linearly dependent, where $f|_{S_a}$ denotes the restriction of f to $S' \cap (S \times \{a\})$.

We need an asymptotic variant of the previous theorem.

Theorem 16. *Let $g : P \times \mathbb{Z} \rightarrow \Lambda$ be a definable morphism between definable sets, for some P and Λ . Assume that P' is a definable subset of $P \times \mathbb{Z}$. Let \mathbf{f} be a tuple of constructible functions on P' . Then there exists a natural number m with the following property. For every $a_0 \in \mathbb{Z}$ and every $F_1, F_2 \in \text{Loc}_m$ such that F_1 and F_2 have isomorphic residue fields, the following holds: if for each $\lambda \in \Lambda(F_1)$, the tuple of functions $\mathbf{f}_{\lambda, F_1}$ is a_0 -asymptotically linearly dependent, then also for each $\lambda_2 \in \Lambda(F_2)$, the tuple of functions $\mathbf{f}_{\lambda_2, F_2}$ is a_0 -asymptotically linearly dependent.*

Proof. We adapt the proof in [9]. Suppose that \mathbf{f} is an r -tuple. We may replace the components f_i of \mathbf{f} with $f_i 1_{P'}$ on $P \times \mathbb{Z}$ to reduce to the case $P' = P \times \mathbb{Z}$. We may reduce asymptotic linear dependence to the identical vanishing of a determinant. Let

$$S = \{(\lambda, a, X_1, \dots, X_r) \mid (X_1, a), \dots, (X_r, a) \in P' \times_\Lambda \dots \times_\Lambda P', g(X_i, a) = \lambda\} \subset \Lambda \times \mathbb{Z} \times P^r.$$

We have a constructible function d on $\Lambda \times \mathbb{Z} \times P^r$ given by the indicator function of S times

$$(\lambda, a, X_1, \dots, X_r) \mapsto \det f_i((X_j, a)).$$

By [8], there is a constructible function $\text{Iv}(d)$ on $\Lambda \times \mathbb{Z}$ whose zero locus coincides with the locus of identical vanishing of d on $(\Lambda \times \mathbb{Z}) \times P^r$. Let $m \in \mathbb{Z}$ be an integer that is sufficiently large for Corollary 12 applied to $\text{Iv}(d)$ and large enough for the zero locus result from [8] to hold.

Let $F_1, F_2 \in \text{Loc}_m$ have isomorphic residue fields. Choose any $a_0 \in \mathbb{Z}$. Assume that for all $\lambda \in \Lambda(F_1)$, the tuple of functions $\mathbf{f}_{\lambda, F_1}$ is a_0 -asymptotically linearly dependent. Then the determinant d_{F_1} vanishes identically for each element of $A(F_1)$, where $A := \{(\lambda, a) \mid a \geq a_0\}$. The set A is definable. Then $\text{Iv}(d)_{F_1}$ is zero on $A(F_1)$. By Corollary 12, $\text{Iv}(d)_{F_2}$ is zero on $A(F_2)$. Following the preceding steps backwards now for F_2 instead of F_1 , from $\text{Iv}(d)_{F_2}$ back to asymptotic linear dependence, we find that for all $\lambda \in \Lambda(F_2)$, the tuple $\mathbf{f}_{\lambda, F_2}$ is a_0 -asymptotically linearly dependent. \square

5. ENDOSCOPIC MATCHING OF SMOOTH FUNCTIONS

5.1. Constructible Shalika germs. We give a proof of Theorem 1 asserting the existence of constructible Shalika germs on the Lie algebra. The Shalika germs are indexed by nilpotent orbits. The number k of nilpotent orbits can vary according to the nonarchimedean field F and the cocycle $z \in Z(F)$. Since we cannot fix k in advance, this complicates matters, and we give a n -tuple of constructible “Shalika germs” for each n . When, for a given nonarchimedean field, n is larger than the actual number of nilpotent orbits, a set Υ of auxiliary parameters is empty, and the construction yields nothing. When n is smaller than the actual number of nilpotent orbits, we obtain an incomplete collection of germs.

This section uses the following notation. We make fixed choices G^{**} , Σ , \mathcal{F} , and so forth, as above. We work with respect to a fixed cocycle space Z that is used to define a form G of G^{**} and inner form of G^* .

For $n \in \mathbb{N}$, let NF^n be the definable subset of n -tuples of pairs $\Upsilon_i = (N_i, \mathbf{f}_i)$, where Υ_i is a Barbasch-Moy pair, N_i is a nilpotent element in $\mathfrak{g}_{\mathbf{f}_i, 0}$, and such that N_1, \dots, N_n are pairwise non-conjugate.

For any n -tuple $\mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_n)$, we write $\text{NF}_{\mathbf{f}}^n$ for the subset of NF^n consisting of those elements whose n -tuple of second coordinates is \mathbf{f} . For any Barbasch-Moy pair $\Upsilon = (N, \mathbf{f})$, let 1_{Υ} be the characteristic function of $N + \mathfrak{g}_{\mathbf{f}, 0+}$.

Proof of Theorem 1. We form the constructible orbital integrals

$$O(X, \Upsilon) = \int_{O(X)} 1_{\Upsilon} d\mu,$$

for $X \in \mathfrak{g}$ that is nilpotent or regular semisimple, $d\mu = d\mu^{\text{nil}}$ or $d\mu^{\text{rss}}$ as appropriate, and Barbasch-Moy pair Υ . If $\Upsilon \in \text{NF}^n$, we let $O(X, \Upsilon)$ be the n -tuple whose i th coordinate is $O(X, \Upsilon_i)$.

If $\Upsilon = ((N_1, \mathbf{f}_1), \dots) \in \text{NF}^n$, we write $\Theta(\Upsilon)$ for the square matrix with entries

$$O(N_i, \Upsilon_j), \quad \text{for } i, j = 1, \dots, n.$$

Let $\Theta^a(\Upsilon)$ be the adjugate matrix of $\Theta(\Upsilon)$, so that

$$\Theta^a(\Upsilon)\Theta(\Upsilon) = \Theta(\Upsilon)\Theta^a(\Upsilon) = d_n(\Upsilon)I_n, \quad \text{where } d_n(\Upsilon) = \det(\Theta(\Upsilon)).$$

For $X \in \mathfrak{g}^{\text{rss}}$ and $\Upsilon \in \text{NF}^n$, we define the Shalika germs $\Gamma(X, \Upsilon)$ as the n -tuple given by the matrix product

$$\Gamma(X, \Upsilon) = \Theta^a(\Upsilon)O(X, \Upsilon).$$

It then follows directly from the definition of adjugate that orbital integrals have a constructible Shalika germ expansion

$$d_n(\Upsilon)O(X, \Upsilon) = \Theta(\Upsilon)\Gamma(X, \Upsilon),$$

That is, up to a determinant d_n , the orbital integral of X of a collection of test functions indexed by Υ can be expanded in terms of the Shalika germs weighted by the nilpotent orbital integrals of the test functions.

If we specialize the data to a nonarchimedean field F , take $z \in Z(F)$, and choose $n = k$ be the number k of nilpotent orbits of $G_z(F)$, then it follows from Theorem 8 that $\text{NF}^n(F)$ is nonempty, and for all $\Upsilon \in \text{NF}^n(F)$, we have $\det \Theta_F(\Upsilon)$ is nonzero. In fact, by suitable ordering of indices, the matrix is upper triangular with nonzero diagonal entries. We obtain the Shalika germ expansion in its usual form. \square

5.2. Strategy. We now prepare for the proof of Theorem 5. This is the most intricate proof in the article. It will occupy Subsections 5.2 through 5.4.

Let $X_H, \bar{X}_H, \bar{X}_G$ be parameters that appear in the transfer factor. For a given X_H , we let X_G be the image of X_H in $\mathfrak{c}_{\mathfrak{g}^*}$ under the morphism

$$\mathfrak{h} \rightarrow \mathfrak{c}_{\mathfrak{h}} \rightarrow \mathfrak{c}_{\mathfrak{g}^*},$$

and let $O^{st}(X_G)$ be the preimage of X_G in \mathfrak{g} . In what follows, the stable orbit $O^{st}(X_G)$ is assumed always to be derived from the parameter X_H in this manner. As described above, we have invariant motivic measures $d\mu_G^{\text{rss}}$ and $d\mu_H^{\text{rss}}$ on $O^{st}(X_G)$ and $O^{st}(X_H)$.

We define κ -orbital integrals by the following equation:

$$(5) \quad O^\kappa(X_H, \bar{X}_H, \bar{X}_G, f) = \int_{x \in O^{st}(X_G)} \Delta(X_H, x, \bar{X}_H, \bar{X}_G) f(x) d\mu_G^{\text{rss}},$$

where f is a constructible function on \mathfrak{g} and the parameters $(X_H, \bar{X}_H, \bar{X}_G)$ run over the definable set V in Equation 1 (omitting the factor for X_G). Similarly, we define stable orbital integrals on \mathfrak{h} by

$$(6) \quad SO(X_H, f^H) = \int_{x \in O(X_H)} f^H(x) d\mu_H^{\text{rss}},$$

where f^H is a constructible function on \mathfrak{h} .

Waldspurger has proved that the germs of O^κ are linear combinations of the germs of SO in Equation 6 when the field F has characteristic zero [34]. We wish to transfer these linear relations among germs to positive characteristic.

The basic naive strategy is to invoke Theorem 16 directly to transfer these linear relations among germs. Unfortunately, the naive strategy does not work, because the coefficients of the linear relation potentially vanish identically on part of the cocycle space Z , which would mean that we would obtain no information about the endoscopic matching for some of the twisted forms of a reductive group. We note that it is not always possible to isolate definably a single isomorphism class of reductive groups G . See Section 6.

The next (less naive) strategy is to take a product $G = G_1 \times \cdots \times G_r$ over all of the groups up to isomorphism that appear as fibers over the cocycle space Z . On this product, the corresponding cocycle space does in fact determine a single group up to isomorphism. The germs on this product are the products of germs of the factors. We can invoke the theorem to transfer linear relations for this product of groups. Unfortunately, this strategy also fails, because all of the κ -Shalika germs might vanish identically for one of the factors; and when this happens, we cannot derive any information from the product about the other individual factors G_i .

This nonetheless, leads to a strategy that works. Again we take r -fold products of factors, but we enhance our collection of factors to include the stable Shalika germs on the quasi-split endoscopic groups (Equation 7). We know how to choose stable Shalika germs that are nonzero. Indeed the stable Shalika germ of the regular nilpotent class in a quasi-split inner form is nonzero. Thus, we are able to avoid the bad situation where one of the factors vanishes identically. This works.

The proof will use the following elementary facts about tensor products of finite dimensional vector spaces in our analysis of the products of Shalika germs. The second statement may be used to extract a nontrivial linear relation on a single factor from a linear relation on the tensor product.

Lemma 17. *Let V_1, \dots, V_r be finite dimensional vector spaces of dimensions $\dim(V_i) = n_i$.*

- (1) *Let $S_i \subset V_i$ be a finite set of vectors for each $i = 1, \dots, r$. If the product of the cardinalities of the sets S_i is $n_1 \cdots n_r$ and if the tensors $w_{i_1} \otimes \cdots \otimes w_{i_r}$, for $w_{i_j} \in S_j$, span $V_1 \otimes \cdots \otimes V_r$, then for each i , the set S_i is a basis of V_i .*
- (2) *For each $i = 1, \dots, r$, let W_i be a subspace of V_i . Let $v_1 \otimes \cdots \otimes v_r \in W_1 \otimes \cdots \otimes W_r$, with $v_i \in V_i$. If for some i , we have $v_j \neq 0$, for all $j \neq i$, then $v_i \in W_i$.*

Proof. This is elementary multilinear algebra. □

In what follows, to use this lemma, we regularly identify finite dimensional subspaces of $C(U_1 \times \cdots \times U_r)$ of the continuous functions on $U_1 \times \cdots \times U_r$, for various choices of spaces U_i , with corresponding finite dimensional subspaces of the tensor product $C(U_1) \otimes \cdots \otimes C(U_r)$, under the map

$$C(U_1) \otimes \cdots \otimes C(U_r) \rightarrow C(U_1 \times \cdots \times U_r),$$

often without explicit mention. We recall that the orbital integral of $f_1 \otimes \cdots \otimes f_r$ on a product $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ is a product of the corresponding orbital integrals on the factors.

5.3. Parameter sets. We establish notation and various parameters that will be used in the proof of Theorem 5. We make all the fixed choices that are established in Section 2.1. Let $(G, H)_Z$ be a definable reductive group G with definable endoscopic group H over a cocycle space Z . We remain in this context until the end of Section 5.4.

5.3.1. Finiteness of the parameter set. The order in which we construct our parameters is extremely important in the proof. Later, in Subsection 5.3.3, we will make choices of constants $r \in \mathbb{N}$, and r -tuples: k, k', \mathfrak{f} , and \mathfrak{f}' that will be field dependent. Before we do that, in this subsection, we bound these parameters so that they run over finitely many possibilities. That is, we define a finite set Ξ that contains all possible parameter choices

$\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}') \in \Xi$. It is important to work with the full finite set Ξ of possibilities until Subsection 5.3.3.

In this paragraph, we define a finite set Ξ of parameters $\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}')$ by placing boundedness constraints on $r, k, k', \mathfrak{f}, \mathfrak{f}'$. (We do not need sharp bounds in this paragraph; we are simply establishing a finiteness result.) We constrain $r \in \mathbb{N}$ to be at most the lim sup over m of maximum over $F \in \text{Loc}_m$ of r_F , where r_F is the number of isomorphism classes of reductive groups G_z obtained as z runs over $Z(F)$. That is, it should be at most the largest number of isomorphism classes of reductive groups in the family G in large residual characteristic. The r -tuple $k \in \mathbb{N}^r$ has components k_i that are constrained to be at most the largest number of nilpotent orbits there can be in sufficiently large residual characteristic in a Lie algebra of some group in the family G . (See Corollary 9.) We constrain the tuple \mathfrak{f} to be an r -tuple, where each component \mathfrak{f}_i is a k_i -tuple of elements of \mathcal{F}_G . Recall that \mathcal{F}_G is a fixed choice that does not depend on the cocycle space Z . The cardinality of \mathcal{F}_G is finite and field independent. The tuple $k' \in \mathbb{N}^r$ is similarly constrained to have components k'_i that are at most the largest number of nilpotent orbits there can be in sufficiently large residual characteristic in a Lie algebra of some group in the family of endoscopic groups H . The r -tuple \mathfrak{f}' is constrained as \mathfrak{f} , but using k' and \mathcal{F}_H . By imposing finiteness constraints as we have on r, k, k', \mathfrak{f} , and \mathfrak{f}' , the parameters $\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}')$ range over a finite set Ξ of possibilities.

5.3.2. Parameter sets that depend on ξ . Let $\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}') \in \Xi$ be any parameter in Ξ . Until Subsection 5.3.3, the element ξ remains a fixed but arbitrary element of Ξ . We often omit ξ and its (fixed but arbitrary) components r, k, k', \mathfrak{f} , and \mathfrak{f}' from the notation. We will give definable sets P, Λ , a definable morphism $g : P \times \mathbb{Z} \rightarrow \Lambda$, and a tuple of constructible functions \mathbf{f} on $P \times \mathbb{Z}$. Let Z^r be the definable set of r -tuples (z_1, \dots, z_r) , where $z_i \in Z$ and such that z_i and z_j are not cohomologous for $i \neq j$.

The Lie algebras $\mathfrak{h}_z, \mathfrak{g}_z$ depend on a parameter $z \in Z$. For $i = 1, \dots, r$, we have spaces

$$\Lambda_i^1 = \{(z, \bar{X}_H, \bar{X}_G, \Upsilon_G, \Upsilon_H) \mid \\ z \in Z, \bar{X}_H \in \mathfrak{h}_z, \bar{X}_G \in \mathfrak{g}_z, \Upsilon_G \in \text{NF}_{G_z, \mathfrak{f}_i}^{k_i}, \Upsilon_H \in \text{NF}_{H_z, \mathfrak{f}'_i}^{k'_i}, \Upsilon_{H,1} : \text{reg}\}.$$

(We apologize for the nesting of subscripts.) The condition $\Upsilon_{H,1} : \text{reg}$ means that we constrain the nilpotent part N of the Moy-Prasad pair $\Upsilon_{H,1}$ to be a regular element. We write $z(\lambda_i), \Upsilon_G(\lambda_i)$, and so forth for the components of $\lambda_i \in \Lambda_i^1$. For $i = 1, \dots, r$, we set

$$\Lambda^r = \Lambda_\xi^r = \{(\lambda_1, \dots, \lambda_r) \mid \lambda_i \in \Lambda_i^1 \quad (z(\lambda_1), \dots, z(\lambda_r)) \in Z^r\}.$$

If $j \in \mathbb{N}$ and $\epsilon \in \{\pm\}$, write

$$\Upsilon(\lambda_i, (j, \epsilon)) = \begin{cases} \Upsilon_G(\lambda_i)_j, & \text{if } \epsilon = -, \quad j \leq k_i, \\ \Upsilon_H(\lambda_i)_j, & \text{if } \epsilon = +, \quad j \leq k'_i. \end{cases}$$

We define:

$$P_i^1 = \{(X_H, \lambda_i) \mid \lambda_i \in \Lambda_i^1, \quad z = z(\lambda_i), \quad X_H \in \mathfrak{h}_z\}.$$

Also, set:

$$P^r = P_\xi^r = \{(X_H, \lambda) \mid \lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda_\xi^r, \quad X_H = (X_{H,1}, \dots, X_{H,r}), \quad (X_{H,i}, \lambda_i) \in P_i^1\}.$$

We have a projection map $P^r \rightarrow \Lambda^r$ onto the second coordinate.

Set $L_i = L_i^+ \sqcup L_i^-$, where $L_i^- = \{1, \dots, k_i\} \times \{-\}$ and $L_i^+ = \{1, \dots, k'_i\} \times \{+\}$. Set $L = \{\ell = (\ell_1, \dots, \ell_r) \mid \ell_i \in L_i\}$, and $L^+ = \{\ell = (\ell_1, \dots, \ell_r) \mid \ell_i \in L_i^+\}$.

We introduce a parameter $a \in \mathbb{Z}$ to construct germs of functions as follows. Let 1_a , for $a \in \mathbb{Z}$ be a sequence of support functions, defined as the preimage in \mathfrak{g} of indicator functions of small balls around 0 in $\mathfrak{c}_{\mathfrak{g}^*}$ tending to 0 as a tends to infinity. We take the preimage of these balls in $\mathfrak{c}_{\mathfrak{h}}$ and \mathfrak{h} and denote that support sequence 1_a as well, by abuse of notation. We may truncate the orbital integrals using these functions: $1_a SO(X, f^H)$ and $1_a O^\kappa(X_H, \bar{X}_H, \bar{X}_G, f)$. Integrals now have an extra integer parameter $a \in \mathbb{Z}$ that we may use to study a -asymptotic linear dependence.

We define constructible functions. For $(X_H, \lambda) \in P_\xi^r$, for $i = 1, \dots, r$, and $\ell_i \in L_i$, set

$$(7) \quad \psi_{i, \ell_i}(X_{H,i}, \lambda_i) = \begin{cases} O^\kappa(X_{H,i}, \bar{X}_H(\lambda_i), \dots, 1_{\Upsilon(\lambda_i, \ell_i)}), & \text{if } \ell_i \in L_i^- \\ SO(X_{H,i}, 1_{\Upsilon(\lambda_i, \ell_i)}), & \text{if } \ell_i \in L_i^+. \end{cases}$$

We also define a tuple $\mathbf{f} = (\dots, 1_a f_\ell, \dots)$ of constructible functions on $P_\xi^r \times \mathbb{Z}$ indexed by $\ell \in L$:

$$((X_H, \lambda), a) \mapsto 1_a f_\ell(X_H)_\lambda := 1_a \prod_{i=1}^r \psi_{i, \ell_i}(X_{H,i}, \lambda_i).$$

In summary, the tuple \mathbf{f} is indexed by ℓ which is a tuple that, when ℓ is in L^- , enumerates the nilpotent orbits in $\mathfrak{g}_{z(\lambda_1)} \times \dots \times \mathfrak{g}_{z(\lambda_r)}$, and when ℓ is in L^+ , enumerates the stable nilpotent distributions on $\mathfrak{h}_{z(\lambda_1)} \times \dots \times \mathfrak{h}_{z(\lambda_r)}$. The components of \mathbf{f} (the functions f_ℓ) mix the reductive group and its endoscopic group together. Each function f_ℓ is an orbital integral on $\mathfrak{g}'_{z(\lambda_1)} \times \dots \times \mathfrak{g}'_{z(\lambda_r)}$, where each factor $\mathfrak{g}'_{z(\lambda_i)}$ is $\mathfrak{h}_{z(\lambda_i)}$ or $\mathfrak{g}_{z(\lambda_i)}$, depending on which side of the disjoint union the parameter $\ell_i \in L_i = L_i^+ \sqcup L_i^-$ lands. These orbital integrals are evaluated at the $(r$ -tuples of) test functions associated with Barbasch-Moy pairs as in Section 5.1.

The condition $\Upsilon_{H,i,1} : \text{reg}$ (in the definition of Λ_i^1) ensures that one of the nilpotent classes on each endoscopic factor is regular, which will be needed later to make sure that the corresponding stable Shalika germ does not vanish. The mixing of factors $\mathfrak{g}'_{z(\lambda_i)}$ allows us to take the regular nilpotent (nonzero) stable germ from the endoscopic group on every factor but one and an arbitrary κ -germ on the remaining factor in order to isolate a given κ -germ.

5.3.3. Field dependent choices. Recall that $\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}')$ denotes an arbitrary element in the finite set Ξ . For each $\xi \in \Xi$, we have produced definable sets $P = P_\xi^r$, $P' = P \times \mathbb{Z}$, $\Lambda = \Lambda_\xi^r$, a definable morphism $g : P \times \mathbb{Z} \rightarrow \Lambda$, and a tuple of constructible functions \mathbf{f} on P' . For each ξ , we invoke Theorem 16 on these data to obtain a constant $m(\xi) \in \mathbb{N}$ with properties as described in the theorem. Recall that $m(\xi)$ is the parameter that gives the restriction on the residual characteristic in the nonarchimedean field. We let m be the maximum of the constants $m(\xi)$ as ξ runs over the finite set Ξ .

As a result of this discussion, we have a constant m that is not tied to any particular parameter ξ . This constant will be used to satisfy the claim in the statement of Theorem 5 of the existence of a constant m with desirable properties.

Let $F_2 \in \text{Loc}_m$. Choose a second field $F_1 \in \text{Loc}_m$ that has characteristic zero and a residue field isomorphic to that of F_2 . We will use the transfer principle between these two fields.

Now that m , F_1 , and F_2 have been constructed, we make a particular choice of $\xi \in \Xi$. (In particular, from this point forward, $\xi = \xi_{m, F_1, F_2}$ and its components $r, k, k', \mathfrak{f}, \mathfrak{f}'$ will no longer

denote arbitrary elements drawn from a finite set Ξ , but will denote choices of parameters that have been adapted to the specific nonarchimedean context of m , F_1 , and F_2 .)

We start with r . Let r be the maximum for which $Z^r(F_1) \neq \emptyset$. That is, it is the number of nonisomorphic reductive groups in the family of cocycles. This maximum can be expressed as a sentence in the Denef-Pas language and can hence be transferred to F_2 by an Ax-Kochen style transfer principle. Thus, r is also the maximum for which $Z^r(F_2) \neq \emptyset$.

Pick any $z \in Z^r(F_2)$. Write $G_i = G_{z_i}$. Let $k = (k_1, \dots, k_r)$ be a tuple such that k_i is the number of nilpotent orbits in $\mathfrak{g}_{z_i}(F_2)$. Choose a k_i -tuple $\mathfrak{f}_i \in \mathcal{F}_{G_i}^{k_i}$ such that $\text{NF}_{G_i, \mathfrak{f}_i}^{k_i}(F_2) \neq \emptyset$. Let $\mathfrak{f} = (\mathfrak{f}_1, \dots, \mathfrak{f}_r)$.

Finally, we choose parameters k' and \mathfrak{f}' as follows. Pick $z \in Z^r(F_1)$. By a transfer principle, we may assume that z is selected in such a way that $\text{NF}_{G_i, \mathfrak{f}_i}^{k_i}(F_1) \neq \emptyset$, where $G_i = G_{z_i}$, for $i = 1, \dots, r$. Write $H_i = H_{z_i}$. Let $k' = (k'_1, \dots, k'_r)$ be a tuple such that k'_i is the dimension of the space of stable Shalika germs on $\mathfrak{h}_{z_i}(F_1)$. By the definition of stable distribution, this equals the dimension of the space of stable orbital integrals supported on the nilpotent set. For each i , let μ_{ij}^{st} , $j = 1, \dots, k'_i$, be a basis of the stable distributions supported on the nilpotent set of $\mathfrak{h}_{z_i}(F_1)$. We have a Shalika germ expansion

$$SO(X_H, f^H) = \sum_{j=1}^{k'_i} \Gamma_{ij}^{st}(X) \mu_{ij}^{st}(f^H),$$

for compactly supported functions f^H on $\mathfrak{h}_{z_i}(F_1)$ and some stable germs Γ_{ij}^{st} . The germ expansion holds on G -regular semisimple elements in a suitable neighborhood (depending on f^H).

For each $i \leq r$, we define a tuple $\Upsilon_i := (\Upsilon_{i1}, \dots, \Upsilon_{ik'_i}) \in \text{NF}_{H_i}^{k'_i}(F_1)$ as follows. We pick parameters Υ_{ij} such that the functions $1_{\Upsilon_{ij}}$, for $j = 1, \dots, k'_i$, form a dual basis to μ_{ij}^{st} (with i fixed and j variable). (The indicator functions for Moy-Prasad pairs span the space dual to invariant distributions with nilpotent support, hence span the quotient space dual to stably invariant distributions with nilpotent support; hence a subset of the functions forms a dual basis.) We can do this by Theorem 8. Note that there is a unique stable distribution supported on the regular nilpotent set. We may pick the dual space in a way that the nilpotent component of Υ_{i1} is regular.

Let \mathfrak{f}'_i be the tuple of second coordinates of Υ_i . Set $\mathfrak{f}' = (\mathfrak{f}'_1, \dots, \mathfrak{f}'_r)$. We have now fixed the parameter $\xi = (r, k, k', \mathfrak{f}, \mathfrak{f}')$.

5.4. Endoscopic matching of Shalika germs. We now give the proof of Theorem 5, adapted to Lie algebras. In Section 5.5, we show that the proof can be easily adapted to the groups. Here, we work in the context of the previous subsection.

Proof of Theorem 5, adapted to Lie algebras. We return to our earlier notation of z as a parameter running over the definable set Z^r (as opposed to an F -point of that set).

We write $\Lambda^r(F_1) = \Lambda_A \cup \Lambda_B$, where

$$\Lambda_A = \{\lambda \in \Lambda^r(F_1) \mid \exists a_0 \in \mathbb{Z}, (\dots, 1_a f_{\ell, \lambda, F_1}, \dots), \text{ for } \ell \in L^+, \text{ is } a_0\text{-asympt. lin. dep.}\}.$$

and $\Lambda_B = \Lambda^r(F_1) \setminus \Lambda_A$.

We claim that the number a_0 that appears in the definition of Λ_A can be chosen to be independent of $\lambda \in \Lambda_A$. Indeed, there are only finitely many different situations up to isomorphism, and we can simply choose the maximum a_0 among finitely many cases.

For $\lambda \in \Lambda_B$, there does not exist an a_0 for which the given tuple of functions is a_0 -asymptotically dependent. Note that the number of stable distributions with nilpotent support on the product of $\mathfrak{h}_{z(\lambda_i)}(F_1)$ does not depend on $\lambda \in \Lambda^r(F_1)$, since different choices of λ merely permute the factors $\mathfrak{h}_{z(\lambda_i)}(F_1)$. This number is $k'_1 \cdots k'_r$, the cardinality of L^+ . We invoke Lemma 17(1) to see that for all i , the functions

$$(8) \quad 1_a \psi_{i, \ell_i}(\cdot, \lambda_i)_{F_1}, \quad \text{for } \ell_i \in L_i^+,$$

are also not a_0 -asymptotically dependent for any a_0 . Thus, the functions $1_a 1_{Y(\lambda_i, \ell_i), F_1}$ span the space dual to the space of stable nilpotent distributions on $\mathfrak{h}_{z_i(\lambda)}(F_1)$. We use Waldspurger's fundamental result [34] that the κ -orbital integral (over F_1 of characteristic zero) admits smooth matching. This implies that for any test function on any factor $\mathfrak{g}_{z_i(\lambda)}$, there exists a_0 such that the κ -orbital integrals of the test function satisfy an a_0 -asymptotic relation with the functions in Formula 8. If we constrain the test functions to be 1_Y for Moy-Prasad pairs, then as we vary the parameters, there are only finitely many situations up to isomorphism. Thus, we may pick a single a_0 that works for all $\lambda \in \Lambda_B$.

We claim that for all $\bar{\ell} \in L \setminus L^+$, there exists $a_0 \in \mathbb{Z}$, such that for each $\lambda \in \Lambda^r(F_1)$, the tuple of functions $1_{a_0, F_1} f_{\ell, \lambda, F_1}$, indexed by $\ell \in \{\bar{\ell}\} \cup L^+$, is a_0 -asymptotically linearly dependent. We may check this claim separately for Λ_A and Λ_B and take the maximum of the constants a_0 that we obtain. For Λ_A we have already checked that such a constant can be chosen, even without including $\bar{\ell}$. For Λ_B , we have checked that we have an a_0 -asymptotic linear dependency that expresses the function with index $\bar{\ell}_i$ in terms of the others for each $i = 1, \dots, r$. By taking tensor products of the relations, we obtain a linear relation on the tuple of functions indexed by $\ell \in \{\bar{\ell}\} \cup L^+$. This gives the claim.

In fact, we may take $a_0 \in \mathbb{Z}$ to be independent of $\bar{\ell}$, because $\bar{\ell} \in L \setminus L^+$ runs over finitely many possibilities.

We apply Theorem 16 to the morphism $f : P^r \times \mathbb{Z} \rightarrow \Lambda^r$. By the theorem, for each $\bar{\ell} \in L \setminus L^+$, and for all $\lambda \in \Lambda^r(F_2)$, the functions f_{ℓ, λ, F_2} , for $\ell \in \{\bar{\ell}\} \cup L^+$ are a_0 -asymptotically linearly dependent. Let $c_{\bar{\ell}, \ell, \lambda} \in \mathbb{C}$ be the coefficients of a nontrivial linear combination (depending on λ). We write this relation as

$$(9) \quad \sum_{\ell \in \{\bar{\ell}\} \cup L^+} c_{\bar{\ell}, \ell, \lambda} 1_a f_{\ell, \lambda, F_2} = 0 \quad \text{for } a \geq a_0.$$

Note that we may assume that $\text{supp}(1_{a'}) \subseteq \text{supp}(1_a)$ if $a' \geq a$, so that when we pass to a larger $a \in \mathbb{Z}$, a given linear relation still holds on the smaller support. Thus, we may take the coefficients $c_{\bar{\ell}, \ell, \lambda}$ to be independent of a . For each $\bar{\ell}$ and λ , the nontriviality of the relation means that $c_{\bar{\ell}, \ell, \lambda} \neq 0$ for some ℓ .

We claim that there exists $\lambda \in \Lambda^r(F_2)$, such that for all $\bar{\ell} \in L \setminus L^+$, $c_{\bar{\ell}, \bar{\ell}, \lambda} \neq 0$. Otherwise, for all $\lambda \in \Lambda^r(F_2)$, there exists $\bar{\ell} \in L \setminus L^+$ such that $c_{\bar{\ell}, \bar{\ell}, \lambda} = 0$. For these choices, Equation 9 becomes

$$(10) \quad \sum_{\ell \in L^+} c_{\bar{\ell}, \ell, \lambda} 1_a f_{\ell, \lambda, F_2} = 0.$$

This asserts that for all $\lambda \in \Lambda^r(F_2)$, the functions $1_a f_{\ell, \lambda, F_2}$, for $\ell \in L^+$, are linearly dependent for $a \geq a_0$. Applying Theorem 16 again in the opposite direction, to go from dependence on F_2 to linear dependence on F_1 , we find that for all $\lambda' \in \Lambda^r(F_1)$, the functions $1_a f_{\ell, \lambda', F_1}$, for $\ell \in L^+$ are linearly dependent. There exists $\lambda' \in \Lambda^r(F_1)$ such that $z(\lambda') \in Z^r(F_1)$ is the parameter used to construct k' and \mathfrak{f}' . This contradicts the choice of k', \mathfrak{f}' , which were chosen to give linear independence. This gives the claim.

Let $\lambda \in \Lambda^r(F_2)$ be the parameter in the claim. The r -tuples k and \mathfrak{f} were constructed using some different parameter $z \in Z(F_2)$. However, different choices of cocycles merely permute the constants k_i . Hence, the product $k_1 \cdots k_r$ is independent of z . Since $\Upsilon_G(\lambda_i) \in \text{NF}^{k_i} := \text{NF}_{G_i, \mathfrak{f}_i}^{k_i}(F_2)$, where $G_i = G_{z(\lambda_i)}$, and since the NF^{k_i} enforces non-conjugacy conditions on its components, we see $G_i(F_2)$ has at least k_i nilpotent orbits. Since the product $k_1 \cdots k_r$ is fixed, in fact, $G_i(F_2)$ has exactly k_i nilpotent orbits. We conclude that the tuples k and \mathfrak{f} are compatible with the parameter $z(\lambda) \in Z(F_2)$.

We write $\Upsilon_H(\lambda) = (\Upsilon_1, \dots, \Upsilon_r)$. For each $i = 1, \dots, r$, the definition of $\lambda_i \in \Lambda_i^1$ constrains Υ_{i1} to be associated with a regular nilpotent orbit. Thus, the index $(1, +) \in L_i^+$ represents the regular nilpotent orbit in the Lie algebra $\mathfrak{h}_{z_i}(F_2)$ of the endoscopic group.

The nonvanishing $c_{\bar{\ell}, \bar{\ell}, \lambda} \neq 0$, gives for each $\bar{\ell} \in L \setminus L^+$ a linear relation for $1_a f_{\bar{\ell}, \lambda, F_2}$ in terms of products of stable orbital integrals on factors $\mathfrak{h}_{z_i}(F_2)$. In particular, for each $i \in \{1, \dots, r\}$, and for each $j \in \{1, \dots, k_i\}$, we define $\bar{\ell} = \bar{\ell}(i, j) = (\bar{\ell}_1, \dots, \bar{\ell}_r)$, where

$$\bar{\ell}_{i'} = \begin{cases} (1, +), & \text{if } i' \neq i \\ (j, -) & \text{if } i' = i. \end{cases}$$

It is known by Langlands and Shelstad that when suitably normalized, the Shalika germ of the stable regular nilpotent class is identically 1 [26]. In particular, it is nonzero. Considering the linear relation involving $1_a f_{\bar{\ell}, \lambda, F_2}$ as a function of $X_{H,i}$ alone, it gives a nontrivial linear relation between functions $X_{H,i} \mapsto \psi_{i, \bar{\ell}_i}(X_{H,i}, \lambda)$ (with nonzero coefficient) and the stable nilpotent orbital integrals on $\mathfrak{h}_{z_i}(F_2)$. Lemma 17(2) extracts the linear relation on a factor from a linear relation on the tensor product. As we run over all j and i , we obtain the desired matching of all κ -Shalika germs on all reductive groups in fibers over the cocycle space $Z(F_2)$. This proves the theorem. \square

5.5. Adaptation from nilpotence to unipotence. We now adapt the results of the previous two sections to unipotent classes in the group.

Proof of Theorem 4. All of the proofs go through with the following changes. Functions on the Lie algebra (Shalika germs, orbital integrals) are replaced with functions on a neighborhood of the identity element in the group. We use notation $\gamma_G, \gamma_H, \bar{\gamma}_G, \bar{\gamma}_H$, and so forth for elements in the appropriate groups, instead of $X_G, X_H, \bar{X}_G, \bar{X}_H$.

The choice of a -data becomes $a_\alpha = \alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}$ (instead of $\alpha(X)$). For this, we restrict γ to the set of topologically unipotent elements (in large residual characteristic), on which square roots can be extracted.

We may assume that the multiplicative characters that occur in the transfer factor are trivial near the identity element. Near the identity element, we have $\Delta_{\text{III}_2} = 1$.

The Kostant section is replaced with the Steinberg section in the quasi-split inner form. In the construction of transfer factors in [11], the compatibility of the Kostant section with the transfer factor is used, as proved by Kottwitz [22]. In this article we have not used

the compatibility with transfer factors, and we do not need to know whether the Steinberg section is compatible with the transfer factor. In fact, we could replace the explicit Steinberg section with an existential assumption of a section. This is because the transfer factor is independent of the choices made in the quasi-split inner form. \square

Proof of Theorem 5 for groups. The proof already given in Section 5.4 for the Lie algebra is readily adapted to the group.

Nilpotent elements are replaced with unipotent elements. In large residual characteristic, the exponential map is defined on the set of nilpotent elements and gives a bijection between nilpotent classes in the Lie algebra and unipotent elements in the group. The exponential map is polynomial and shows that the set of unipotent elements is a definable set.

The set of parahoric subalgebras is replaced with the set of parahoric subgroups. We use Theorem 10 for the properties of the Barbasch-Moy pairs. We have structured our proofs in the nilpotent case in such a way that the proofs in the unipotent case apply with minimal changes. \square

5.6. Endoscopic matching of smooth functions on the group. In this section, we prove Theorem 6.

In [26], Langlands and Shelstad define the notion of matching (transfer) of orbital integrals from a reductive group over a nonarchimedean field to an endoscopic group. In particular, they introduce the transfer factors that are used for the matching. In [24], they take the further step of reducing the existence of matching to a local statement at the identity in the centralizer of a semisimple element. We review this reduction below. It is this reduction that we will use in this section, in combination with the endoscopic matching of Shalika germs from the previous section, to establish endoscopic matching in sufficiently large characteristic.

In earlier sections, the primary focus was on constructible functions. In this section, the focus is the p -adic theory.

We recall some results from [24]. That article assumes that F is a nonarchimedean field of characteristic zero. We need to relax the restriction on the characteristic to allow fields of large positive characteristic. We briefly sketch how the assumption on the characteristic enters into the article [24] and how it can be relaxed. The assumption of characteristic zero is used in the parts of the article devoted to the archimedean places. These sections of the article do not concern us here. The assumption is used in an essential way in [24, §6.6 Case 3] in a global argument used to deal with fields of even residual characteristic. Since we are assuming large residual characteristic, we may disregard that case. The complete reducibility of a module is used [24, §5.5]. Again, it is enough to assume the characteristic is large. The cocycle calculations, which form the bulk of the article, do not require the assumption that the field has characteristic zero.

That article also cites other sources that assume that the characteristic is zero. We need Harish-Chandra's descent of orbital integrals near a semisimple element, which holds in positive characteristic by [4, §7]. They apply descent to Harish-Chandra character of a representation; but as they point out, it applies in fact to any G -invariant distribution. Their result relies on Harish-Chandra's submersion principle, which in positive characteristic is proved by Prasad in [2, Appendix B]. The article cites [26], which also assumes the field has characteristic zero. But here again, the arguments are readily adapted to large characteristics.

Let G be a reductive group over a nonarchimedean field F , and let H be an endoscopic group of G . We recall [24, §2.1] that (G, H) admits *endoscopic Δ -matching* if for each $f \in C_c^\infty(G(F))$ there exists $f^{\tilde{H}} \in C_c^\infty(\tilde{H}(F), \theta)$ such that f and $f^{\tilde{H}}$ have⁴ Δ -matching orbital integrals:

$$SO(\tilde{\gamma}_H, f^{\tilde{H}}) = \sum_{\gamma_G} \Delta(\tilde{\gamma}_H, \gamma_G) O(\gamma_G, f)$$

for all strongly G -regular $\tilde{\gamma}_H \in \tilde{H}(F)$. The sum is finite, running over representatives γ_G of regular semisimple conjugacy classes such that $\Delta(\tilde{\gamma}_H, \gamma_G) \neq 0$. The group \tilde{H} is obtained as an admissible z -extension from H as explained in [26, §4.4]. It has an associated multiplicative character θ on $Z(H)(F)$.

We recall that (G, H) admits *local endoscopic Δ -matching* if for any $f \in C_c^\infty(G(F))$ we can find $f^H \in C_c^\infty(H(F))$ such that

$$SO(\gamma_H, f^H) = \sum_{\gamma_G} \Delta_{loc}(\gamma_H, \gamma_G) O(\gamma_G, f)$$

for all strongly G -regular elements γ_H near 1 in $H(F)$. We allow the size of the neighborhood of 1 to depend on f^H . In fact, it is enough to match finitely many functions that span the dual to the space of invariant distributions supported on the unipotent set.

For each semisimple element $\epsilon_H \in H(F)$ that is the image of some $\epsilon_G \in G(F)$, Langlands and Shelstad construct an endoscopic pair $(G_{\epsilon_G}, H_{\epsilon_H})$ with corresponding transfer factor Δ_ϵ . The group G_{ϵ_G} is the connected centralizer of ϵ_G .

We will need the following result from [24, 2.3.A].

Theorem 18 (Langlands-Shelstad). *Suppose all pairs $(G_{\epsilon_G}, H_{\epsilon_H})$ have local endoscopic Δ_ϵ -matching, then (G, H) has endoscopic Δ -matching.*

Conversely, if endoscopic matching holds on the entire group, then local endoscopic matching holds in particular at the identity: $\epsilon_G = 1$ and $\epsilon_H = 1$.

Note that the admissible z -extensions are needed to formulate the statement of endoscopic Δ -matching, but they do not appear in the statement of local endoscopic matching. Thus, the admissible z -extensions will play no part in our proof.

Proof of Theorem 6. Let G be a definable reductive group over a cocycle space Z .

Over a nonarchimedean field, there are only a finite number of endoscopic groups, up to conjugacy. Also, there are only finitely many different centralizers that are obtained by descent, up to conjugacy [24, §2.2]. We can state this uniformly as the field varies: there are finitely many definable connected reductive centralizers each having finitely many definable reductive endoscopic groups such that for all fields of sufficiently large residual characteristic, all the centralizers and their endoscopic groups are obtained from these finitely many definable groups by specialization to the field in question up to conjugation of centralizers and equivalence of endoscopic data. For each separate definable pair $(G_{\epsilon_G}, H_{\epsilon_H})$, we will

⁴Langlands and Shelstad fix $\tilde{\gamma}_H$ and $\bar{\gamma}_G$ in the transfer factor and then drop them from notation, so that only the first two variables of Δ are displayed. Throughout most of [26], they assume that the split extension of \hat{H} by W_F (the Weil group of F) is an L -group (§1.2). When they drop the assumption in §4.4, the general transfer factor is denoted Δ and the local factor becomes denoted Δ_{loc} . We follow their conventions here.

obtain a natural number m such that our arguments work for all fields $F \in \text{Loc}_m$. Then the maximum of all such m will work for all descent data.

By Theorem 5, for each endoscopic pair $(G_{\epsilon_G}, H_{\epsilon_H})$, there exists m such that we have local endoscopic Δ_ϵ matching for all $F \in \text{Loc}_m$. Note that the local endoscopic matching is a direct consequence of the endoscopic matching of Shalika germs. This completes the proof. \square

6. CLASSIFICATION OF DEFINABLE REDUCTIVE GROUPS

We rely on the classification of reductive groups from [16], [33], [31], [29], [30], closely following the presentation in [17]. All reductive groups in this section are assumed to split over a tamely ramified field extension. We recall that we do not have access to a Frobenius element of the Galois group, which is used in essential ways in the classification. We obtain a slightly weaker classification using qFr , a fixed generator of Σ^{unr} .

We describe the extent to which nonisomorphic reductive groups over a given field F can have the same set of fixed choices as described in Section 2.1. For simplicity, assume that G and G' are forms of a simple, simply connected, split group G^{**} over F . Assume that G and G' have identical fixed choices. There are obvious cases when this occurs:

- (1) G and G' are inner forms of $SL(n)$, and their invariants in the Brauer group have equal denominators, when expressed in lowest terms.
- (2) G and G' have isomorphic split outer forms, are not split, are quasi-split, and split over different ramified extensions of the same degree. For example, we may have two quasi-split special unitary groups that split over different ramified quadratic extensions.
- (3) G and G' are not quasi-split, but are inner forms of quasi-split groups G^* and G'^* described in the previous situation (and that split over ramified quadratic extensions). For example, we may have two nontrivial inner forms of quasi-split special unitary groups that split over different ramified quadratic extensions.

Remark 19. In the second case, the group of automorphisms of a connected Dynkin diagram, when nontrivial, is cyclic of prime degree except in the case of D_4 , which has automorphism group S_3 . From this it follows that different tamely ramified extensions of the same degree have isomorphic Galois groups and isomorphic inertia subgroups.

Lemma 20. *Let G and G' be forms of a simple, simply connected split group G^{**} over a nonarchimedean field F , for which the fixed choices can be chosen to be equal. Assume the groups split over a tamely ramified extension of F . Then G and G' fall into one of the three cases just enumerated.*

We remark that the classification in this lemma with respect to specific fixed data is not known to coincide with the classification with respect to arbitrary definable data. This is a question we leave unanswered.

Proof. Specifically, suppose that we are given F and an enumerated Galois group $\text{Gal}(K/F)$ that splits G . Suppose that it has a short exact sequence

$$1 \rightarrow \text{Gal}(K/E) \rightarrow \text{Gal}(K/F) \rightarrow \text{Gal}(E/F) \rightarrow 1,$$

where E is the maximal unramified extension of F in K , that is isomorphic with the fixed data

$$1 \rightarrow \Sigma^t \rightarrow \Sigma \rightarrow \Sigma^{unr} \rightarrow 1.$$

The enumeration fixes a distinguished generator qFr of Σ^{unr} , which need not correspond with the Frobenius generator of $\text{Gal}(E/F)$.

In what follows, we use the notation of the fixed choices in Subsection 2.1. We refer to [17] for a detailed treatment of this subject material.

First, consider the case $e = 1$, where $e = [\Sigma : \Sigma^t]$. This means that the group splits over an unramified extension. Part of the fixed data is an action $\phi : \Sigma^{unr} \rightarrow \text{Aut}(\mathcal{R})$ on the affine diagram. If the diagram is not \mathcal{A}_n , then $\phi(\text{qFr})$ is determined by its order up to an automorphism of \mathcal{R} . Thus, by adjusting by an automorphism, we may assume that $\phi(\text{qFr}) = \phi(\text{Fr})$, where Fr is the Frobenius automorphism of $\text{Gal}(E/F)$, under an identification of the Galois group with Σ^{unr} . We then know the action of Frobenius on \mathcal{R} , which is enough to determine the group G up to isomorphism. Thus the fixed data leaves no flexibility in the isomorphism class of the group.

When $e = 1$ and $\mathcal{R} = \mathcal{A}_n$, the image of ϕ determines a choice of Tits index (or equivalently to the set of simple roots specifying the minimal parabolic subgroup). We can specify the minimal parabolic but not the particular form of the anisotropic kernel (unless we are given the Frobenius generator). In other words, fixing the group $\phi(\Sigma^{unr})$ rather than $\phi(\text{Fr})$ is sufficient to determine the denominator of the invariant in the Brauer group, but not the numerator. This is the first case.

Now consider the case $e > 1$. These are groups that split over a ramified extension. The fixed data determines whether the group is quasi-split. The quasi-split groups are precisely those for which $\phi(\text{qFr}) = 1$ (that is, the case $c = 1$ in [17, §7]).

Assume $e > 1$ and the group is quasi-split. In this case, the fixed data gives

$$\rho_G : \Sigma \rightarrow \text{Aut}(\text{Dyn}).$$

The image of Σ is cyclic with corresponding extension totally ramified, except in the case of an S_3 -action on the Dynkin diagram. This homomorphism and the identification of the short exact sequence for $\text{Gal}(K/F)$ with that for Σ determine the group up to isomorphism. See Remark 19. This is the second case.

Finally, consider the case of non quasi-split forms with $e > 1$. In this situation, the only affine diagrams ${}^e\mathcal{R}$ with automorphisms are ${}^2\mathcal{A}_{2n+1}$ (with one inner form) and ${}^2\mathcal{D}_n$ (with one inner form). There is a unique inner form, so it is necessarily uniquely determined by the fixed data, once the ramified quadratic splitting field K/F is given. This is the third case. \square

7. OPEN PROBLEMS

7.1. Igusa theory. Langlands and Igusa have results about the asymptotic behavior of integrals [25] [20]. Under certain assumptions, a family of p -adic integrals with parameter x in a nonarchimedean field F can be expanded in a finite asymptotic series

$$(11) \quad \sum_{a,b,\theta} c_{a,b,\theta} \theta(x) |x|^a (\text{ord } x)^b$$

for some integers a, b , multiplicative characters $\theta : F^\times \rightarrow \mathbb{C}^\times$, and for some constants $c_{a,b,\theta}$. The constants are given as principal-valued integrals. The proof of the asymptotic series relies on the Igusa zeta function.

In the article, we have presented some results about the asymptotic behavior of integrals. This suggests that it may be possible to adapt the asymptotic series expansion (Equation 11) to a motivic context. A general motivic theory of multiplicative characters is not yet available. Nonetheless, an inspection of the proofs in Langlands suggest that it might be possible to obtain an expansion similar to (Equation 11) using far less than a general theory of multiplicative characters.

7.2. Twisted and nonstandard endoscopy. In this article, we have transferred the endoscopic matching of smooth functions for standard endoscopy. Kottwitz, Shelstad, Waldspurger, and Ngô have developed a corresponding theory for twisted endoscopy [23], [35], [28]. We expect that these results can be combined with our methods to obtain transfer results for twisted endoscopic matching of smooth functions.

7.3. Smooth transfer and the trace formula. Our article relies on Waldspurger's proof that the fundamental lemma implies smooth matching in characteristic zero. Waldspurger's primary tool is a global Lie algebra trace formula based on the adelic Poisson summation formula on the Lie algebra. Hrushovski-Kazhdan and Chambert-Loir-Loeser have developed motivic Poisson summation formulas [18], [7]. In fact, Hrushovski and Kazhdan have applied the motivic Poisson summation formula to show that if two local test functions on two division algebras have matching orbital integrals, then their Fourier transforms also have matching orbital integrals [18, Theorem 1.1]. Their methods and result are closely related to Waldspurger's smooth matching result. This suggests the problem of adapting Waldspurger's proof to a motivic setting.

7.4. Definably indistinguishable reductive groups. In Section 6, we described the extent to which fixed choices may be used to distinguish various isomorphism classes of reductive groups. We may ask further whether *groups in the same family (1), (2), (3) of Section 6 are indistinguishable with respect to all (first-order) properties expressible in the Denef-Pas language?*

We may also ask *to what extent is the harmonic analysis on two groups in the same family the same?*

We give an example. Consider the Denef-Pas language with constants in $\mathbb{Z}[[\pi]]$, where π is simply a variable. Consider the quasi-split definable simply connected group of type ${}^e A_n$, ${}^e D_n$, or ${}^e E_6$, with $e = 2$ or 3 as appropriate, associated with the totally ramified extension K/VF of degree e :

$$K = \text{VF}[t]/(t^e - \pi), \quad \text{ord}(\pi) = 1.$$

We make the remarkable observation that the given definable group has nonisomorphic interpretations as reductive groups over a nonarchimedean field F , simply because nonisomorphic field extensions of F are obtained by different choices of the uniformizer $\pi_F \in F$ interpreting π . So in particular, for two different quasi-split unitary groups over F , splitting over different ramified quadratic extensions, the Shalika germs are given by identical constructible formulas, differing over F only by the specialization of π to different uniformizers $\pi_F \in F$.

This suggests that the harmonic analyses on two groups in the same family are very closely related.

7.5. Definability of nilpotent orbits. We have carefully avoided the issue of the definability of nilpotent orbits in this article. This has required us to use certain roundabout constructions such as the set NF_G^k .

If we allow a free parameter N that ranges over the nilpotent set, then the nilpotent orbits are trivially definable by the formula

$$O(N) = \{N' \mid \exists g \in G, \operatorname{ad} g(N) = N'\}.$$

However, this leaves open the question of whether nilpotent orbits can be defined in the Denef-Pas language without the use of parameters. Diwadkar has obtained partial results on this problem [15]. The results of Barbasch-Moy and DeBacker on the classification of nilpotent orbits –together with definability for the Moy-Prasad filtration [10] – reduce this problem to the definability of nilpotent orbits (or more simply of distinguished nilpotent orbits) in reductive groups over finite fields. For some groups such as $SL(n)$, roots of unity are required to specify the nilpotent orbits. But in the adjoint case, the situation is simpler. This leads to the following question: Let $G = G_{\operatorname{adj}}$ be an adjoint semisimple linear algebraic group defined over a finite field. When the characteristic is sufficiently large, is each nilpotent orbit in the Lie algebra definable in the first-order language of rings?

8. ERRATA

We mention two corrections to [11]. In Section 3.2 of that article, the parameter space of unramified extensions requires more than the single parameter a that was given. This does not affect the correctness of the results, but the dimension of the parameter space needs to be changed throughout the article.

In Section 4.2, the construction of the measure on the stable orbit as a Leray residue is correct, but the surrounding comments contain inaccuracies and should be deleted.

REFERENCES

- [1] Jeffrey D. Adler, *Refined anisotropic K -types and supercuspidal representations*, Pacific J. Math **185** (1998), no. 1.
- [2] Jeffrey D. Adler and Stephen DeBacker, *Murnaghan–Kirillov theory for supercuspidal representations of tame general linear groups, with appendices by Reid Huntzinger and Gopal Prasad*, J. Reine Angew. Math. **575** (2004), 1–35. MR2097545 (2005j:22008)
- [3] ———, *Some applications of Bruhat–Tits theory to harmonic analysis on the Lie algebra of a reductive p -adic group*, Michigan Math. J. **50** (2002), no. 2, 263–286. MR1914065 (2003g:22016)
- [4] Jeffrey D. Adler and Jonathan Korman, *The local character expansion near a tame, semisimple element*, Amer. J. Math. **129** (2007), no. 2, 381–403.
- [5] Dan Barbasch and Allen Moy, *Local character expansions*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 5, 553–567 (English, with English and French summaries). MR1474804 (99j:22021)
- [6] Roger William Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Vol. 20, Wiley New York, 1985.
- [7] Antoine Chambert-Loir and François Loeser, *Motivic height zeta functions*, arXiv preprint arXiv:1302.2077 (2013).
- [8] Raf Cluckers, Julia Gordon, and Immanuel Halupczok, *Integrability of oscillatory functions on local fields: transfer principles*, Duke Math. J. **163** (2014), no. 8, 1549–1600.

- [9] ———, *Transfer principles for bounds of motivic exponential functions*, preprint arXiv:1412.4573 (2014).
- [10] ———, *Local integrability results in harmonic analysis on reductive groups in large positive characteristic*, Ann. Sci. Éc. Norm. Sup **47** (2014), no. 6, 1163–1195.
- [11] Raf Cluckers, Thomas Hales, and François Loeser, *Transfer principle for the fundamental lemma*, On the stabilization of the trace formula, 2011.
- [12] Raf Cluckers and François Loeser, *Constructible motivic functions and motivic integration*, Inventiones mathematicae **173** (2008), no. 1, 23–121.
- [13] ———, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Ann. of Math. **171** (2010), no. 2, 1011–1065.
- [14] Stephen DeBacker, *Parametrizing nilpotent orbits via Bruhat–Tits theory*, Ann. of Math. **156** (2002), no. 1, 295–332. MR1935848 (2003i:20086)
- [15] Jyotsna Mainkar Diwadkar, *Nilpotent conjugacy classes of reductive p -adic Lie algebras and definability in Pas’s language*, Ph.D. Thesis, 2006.
- [16] Philippe Gille, *Rationality properties of linear algebraic groups and Galois cohomology*, MCM (2007).
- [17] Benedict H Gross, *Parahorics* (2012). preprint.
- [18] Ehud Hrushovski and David Kazhdan, *Motivic Poisson summation*, Mosc. Math. J **9** (2009), no. 3, 569–623.
- [19] James E Humphreys, *Linear algebraic groups*, New York (1975).
- [20] Jun-ichi Igusa, *Forms of higher degree*, Research Lectures on Mathematics and Physics, vol. 59, Tata Institute, 1978.
- [21] Ju-Lee Kim, *Hecke algebras of classical groups over p -adic fields and supercuspidal representations*, Amer. J. Math. **121** (1999), 967–1029.
- [22] Robert Kottwitz, *Transfer factors for Lie algebras*, Representation Theory of the AMS **3** (1999), no. 6, 127–138.
- [23] Robert E Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999).
- [24] R Langlands and Diana Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, 1990, pp. 485–563.
- [25] Robert P Langlands, *Orbital integrals on forms of $SL(3)$, I*, Amer. J. Math. (1983), 465–506.
- [26] Robert P Langlands and Diana Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1, 219–271.
- [27] George McNinch, *Nilpotent orbits over ground fields of good characteristic*, Math. Ann. **329** (2004), 49–85.
- [28] Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publications mathématiques de l’IHÉS **111** (2010), no. 1, 1–169.
- [29] Victor Petrov and Anastasia Stavrova, *The Tits indices over semilocal rings*, Transformation groups **16** (2011), no. 1, 193–217.
- [30] Mark Reeder, *Torsion automorphisms of simple Lie algebras*, Enseign. Math.(2) **56** (2010), no. 1-2, 3–47.
- [31] Martin Selbach, *Klassifikationstheorie halbeinfacher algebraischer Gruppen* (1976). unpublished.
- [32] Joseph A Shalika, *A theorem on semi-simple p -adic groups*, Ann. of Math. (1972), 226–242.
- [33] Jacques Tits, *Classification of algebraic semisimple groups*, Algebraic groups and discontinuous subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), 1966, pp. 33–62.
- [34] J-L Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Mathematica **105** (1997), no. 02, 153–236.
- [35] Jean-Loup Waldspurger, *L’endoscopie tordue n’est pas si tordue*, AMS, 2008.